Post-Keplerian effects in binary systems

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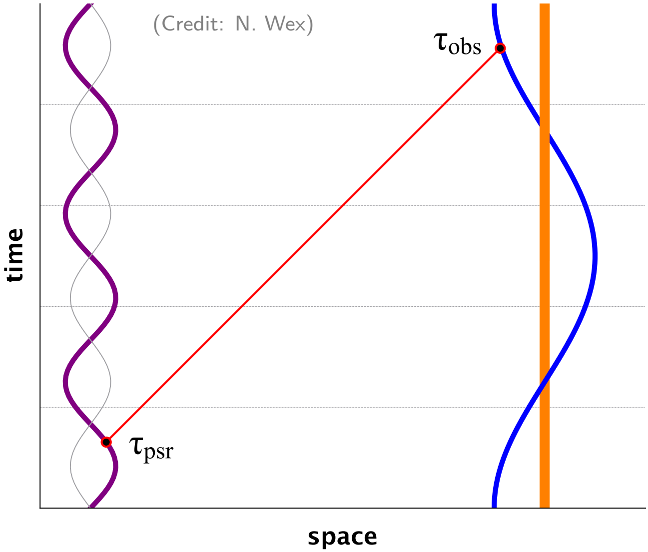
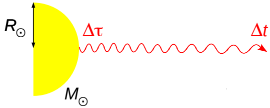
The problem of binary pulsar timing



Some classical tests of General Relativity

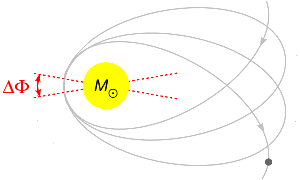
- Gravitational redshift of light

$$\frac{\Delta\tau}{\Delta t} = 1 - \frac{G}{c^2}\frac{M_\odot}{R_\odot}$$



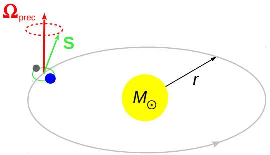
- Perihelion advance of Mercury

$$\Delta\Phi = \frac{G}{c^2}\frac{6\pi M_\odot}{a\,(1-e^2)}$$



- Precession of Earth-Moon spin

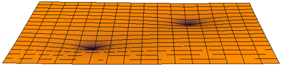
$$\boldsymbol{\Omega}_{\text{prec}} = \frac{G}{c^2}\,\mathbf{v}\times\boldsymbol{\nabla}\left(\frac{3M_\odot}{2r}\right)$$



(Credit: N. Wex)

Different regimes of gravity

quasi-stationary,
weak-field regime



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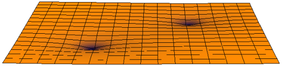

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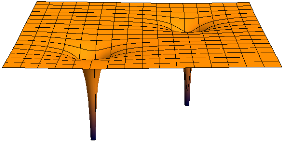

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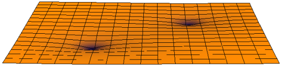

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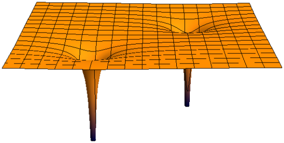
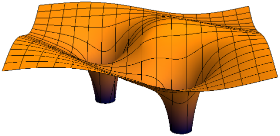


highly dynamical,
strong-field regime



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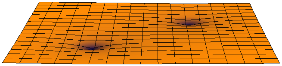

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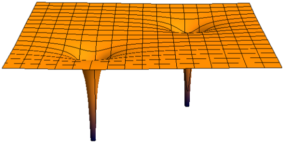

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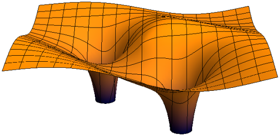
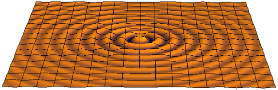


radiative regime



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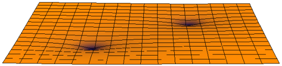

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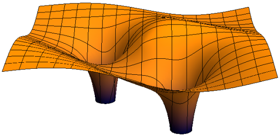

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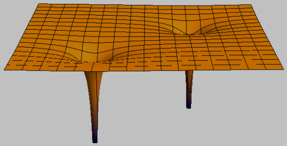

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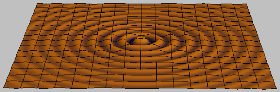
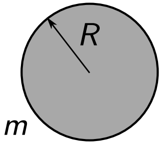


Effacement of the internal structure



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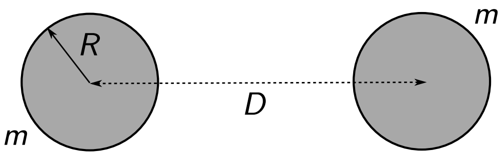

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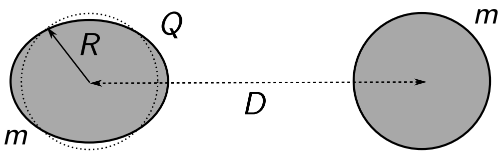

- Induced quadrupole moment: $Q_{ij} \sim R^5 \, \partial_i \partial_j U \sim R^5 (Gm/D^3)$

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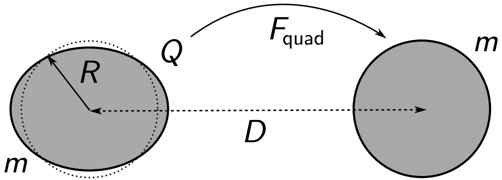

- Induced quadrupole moment: $Q_{ij} \sim R^5\, \partial_i \partial_j U \sim R^5 (Gm/D^3)$
- Induced quadrupolar force: $F^i_{\rm quad} \sim m\, \partial_i \big(\frac{Q}{r^3} \big) \sim R^5 (Gm^2/D^7)$

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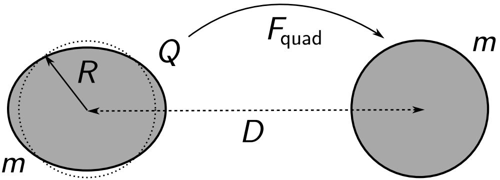

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- For a compact body with $R \sim Gm/c^2$,

$$\frac{F_{\rm quad}}{F_{\rm Newt}} \sim \frac{(G^6/c^{10})(m/D)^7}{Gm^2/D^2} \sim \left(\frac{Gm}{c^2 D}\right)^5 \sim \left(\frac{v}{c}\right)^{10} \ll 1$$

Outline

1. The Newtonian two-body problem
2. Relativistic celestial mechanics
   - Periastron advance
   - Geodetic spin precession
   - Gravitational radiation reaction
3. Light propagation in curved spacetime
   - Einstein delay
   - Shapiro delay
4. Testing relativistic gravitation

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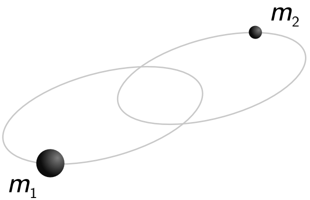
Real two-body problem

$$H(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{Gm_1 m_2}{r_{12}}$$



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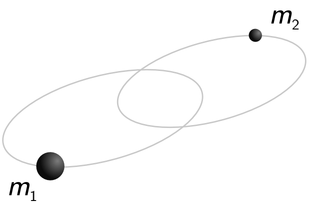

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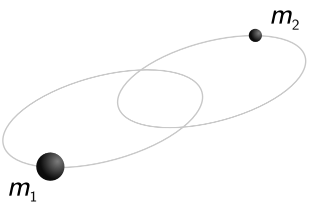
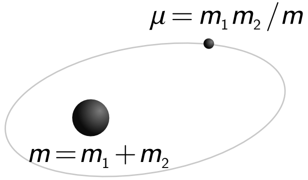


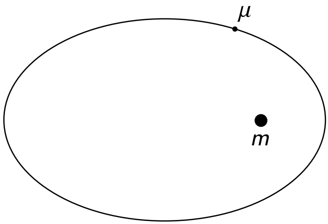
$$\Big\downarrow \quad \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}$$

Effective one-body problem

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2\mu} - \frac{Gm\mu}{r}$$



Keplerian solution for bound orbits



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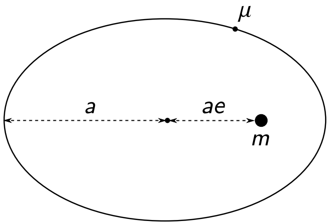
Keplerian elements

- eccentricity e
- semi-major axis a



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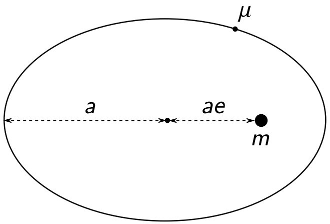
Constants of motion

- energy $E = -Gm\mu/(2a)$
- ang. mom. $L = \mu\sqrt{Gma(1-e^2)}$



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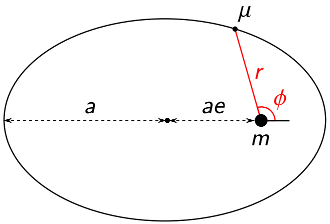
Parametric solution

$$r(u) = a\,(1 - e\cos u)$$

$$\phi(u) = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right)$$

$$\nu(u) = u - e\sin u = n\,(t - t_0)$$



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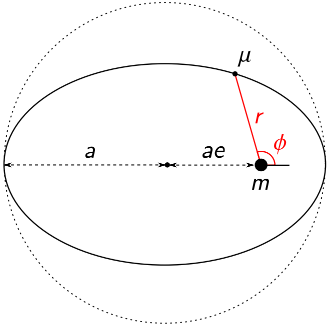

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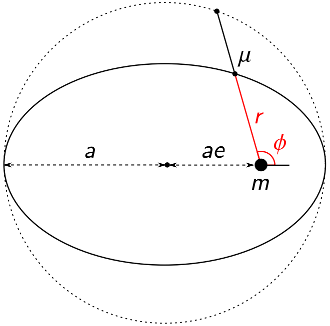

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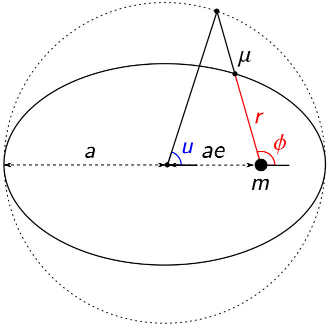

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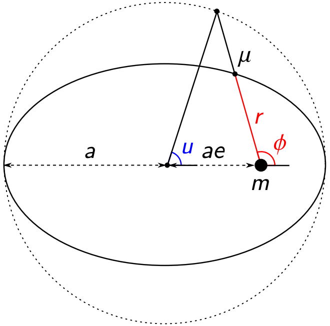
Kepler's third law: $n^2 a^3 = Gm$



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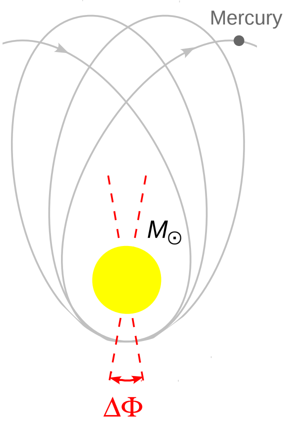
Relativistic perihelion advance of Mercury

Leading-order relativistic angular advance per orbit:

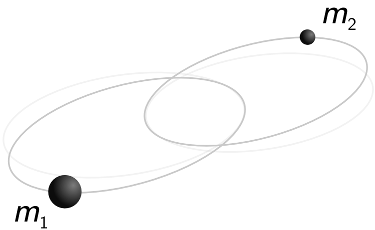
$$\Delta\Phi = \frac{6\pi G M_\odot}{c^2 a\,(1-e^2)}$$

Origin	**Amplitude ($''$/cent.)**
Other planets	531.63 ± 0.69
Sun oblateness	0.028 ± 0.001
General relativity	42.98 ± 0.04
Total	574.64 ± 0.69
Observed	574.10 ± 0.65



Two-body Hamiltonian at 1PN order

$$H = H_{\mathrm{N}} + \frac{1}{c^2} H_{1\mathrm{PN}} + \mathcal{O}(c^{-4})$$



$$H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{4}\frac{G^2 m_1 m_2 m}{r_{12}^2} + \frac{1}{8}\frac{G m_1 m_2}{r_{12}}\left[-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1 m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right]$$

Periastron advance in binary pulsars

- Generic eccentric orbit parametrized by the *two* frequencies

$$n = \frac{2\pi}{P}\,, \quad \langle\dot{\varphi}\rangle = \frac{2\pi + \Delta\Phi}{P}$$

- Leading-order periastron advance:

$$\dot{\omega} \equiv \frac{\Delta\Phi}{2\pi} = \frac{3G(m_1+m_2)}{c^2 a\,(1-e^2)}$$

PSR	**Amplitude (°/year)**
B1913+16	4.226598 ± 0.000005
J0737-3039	16.89947 ± 0.00068

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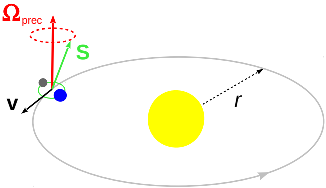
Geodetic spin precession

[de Sitter, MNRAS 1916]

- GR predicts that a test spin $\mathbf{S}$ with velocity $\mathbf{v}$ in a gravitational potential $\Phi \equiv GM/(c^2 r)$ will precess according to

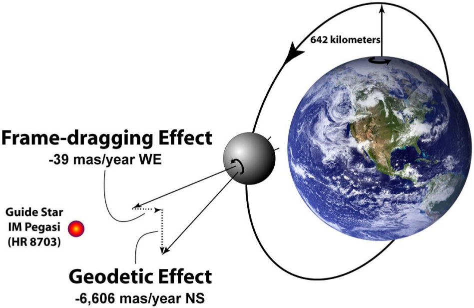
$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = \boldsymbol{\Omega}_{\text{prec}} \times \mathbf{S} \quad \text{where} \quad \boldsymbol{\Omega}_{\text{prec}} \simeq \frac{3}{2}\,\mathbf{v} \times \boldsymbol{\nabla}\Phi$$

- Precession of Earth-Moon spin axis of $\sim 1.9''$/cent. confirmed using Lunar Laser Ranging [Shapiro *et al.*, PRL 1988]



Geodetic effect in Gravity Probe B

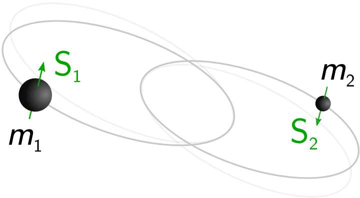
[Everitt *et al.*, PRL 2011]



	Measured	**Predicted**
Geodetic precession (mas/yr)	6602 ± 18	6606
Frame-dragging (mas/yr)	37.2 ± 7.2	39.2

Spin-orbit coupling at leading order

[Barker & O'Connell, PRD 1975]



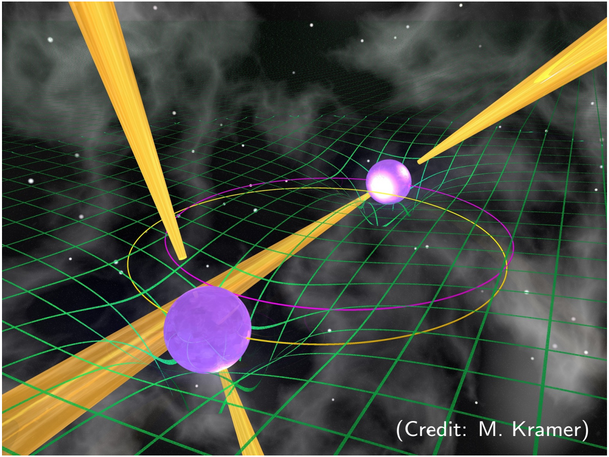
$$\frac{\mathrm{d}\mathbf{S}_a}{\mathrm{d}t} = \boldsymbol{\Omega}_a \times \mathbf{S}_a$$

$$H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = H_{\text{orb}}(\mathbf{x}_a, \mathbf{p}_a) + \overbrace{\sum_b \boldsymbol{\Omega}_b(\mathbf{x}_a, \mathbf{p}_a) \cdot \mathbf{S}_b}^{\text{spin-orbit coupling}}$$

$$\boldsymbol{\Omega}_1(\mathbf{x}_a, \mathbf{p}_a) = \frac{G}{c^2 r_{12}^2} \left(\frac{3m_2}{2m_1} \mathbf{n}_{12} \times \mathbf{p}_1 - 2\mathbf{n}_{12} \times \mathbf{p}_2 \right) \propto \mathbf{L}$$

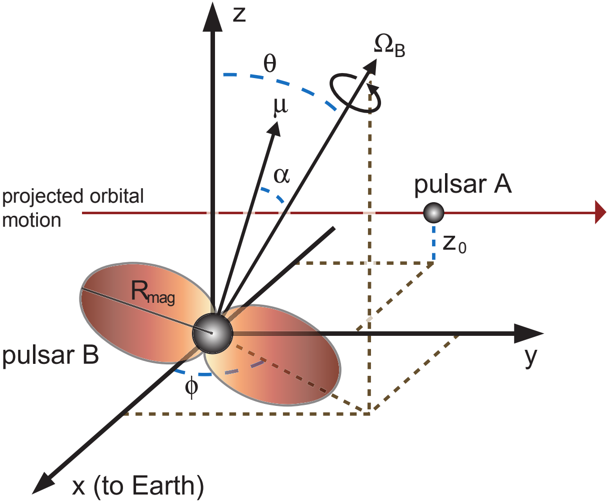
The double pulsar PSR J0737-3039

[Burgay *et al.*, Nature 2003]



Spin precession of pulsar B

[Breton *et al.*, Science 2008]



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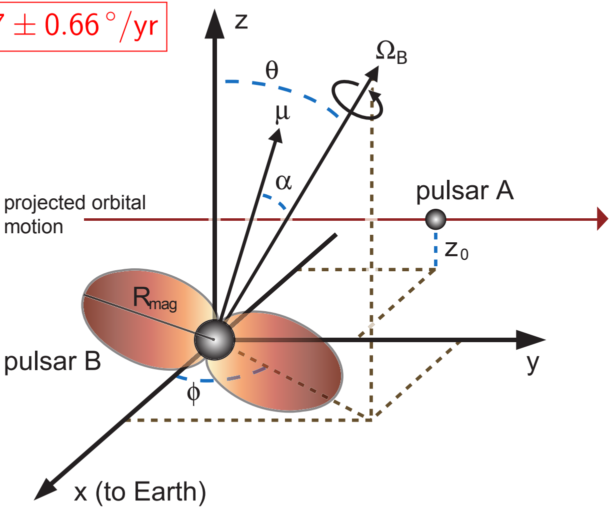
$$\Omega_B = 4.77 \pm 0.66\,^\circ/\mathrm{yr}$$



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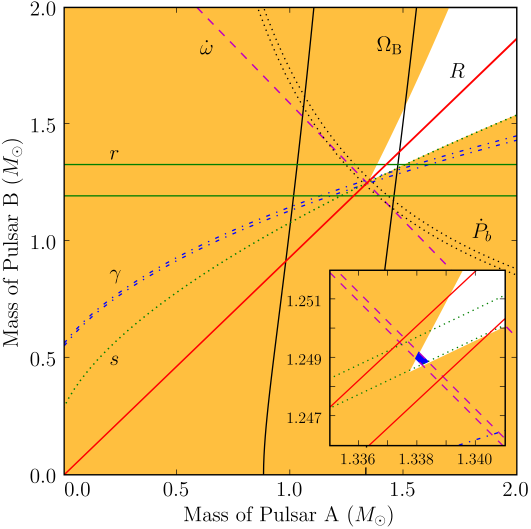
[Breton *et al.*, Science 2008]



Outline

1. The Newtonian two-body problem
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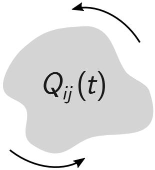
Einstein quadrupole formula



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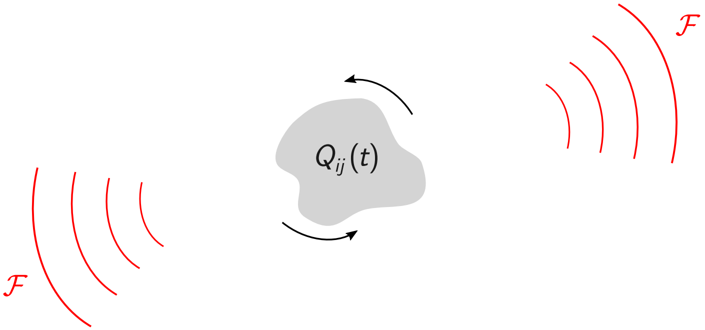
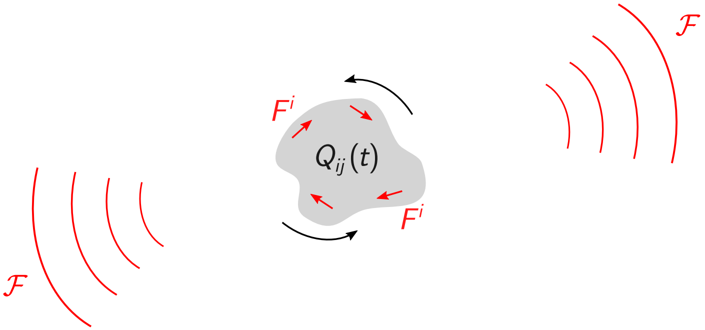


Gravitat. wave energy flux

$$\mathcal{F}(t) = \frac{G}{5c^5} \left\langle \dddot{Q}_{ij} \dddot{Q}_{ij} \right\rangle$$

Einstein quadrupole formula



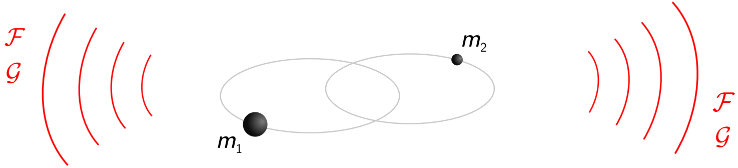
Gravitat. wave energy flux

Radiation reaction force

$$\mathcal{F}(t) = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij} \right\rangle \qquad \longleftrightarrow \qquad F^i(t,\vec{x}) = \frac{2G}{5c^5} Q^{(5)}_{ij} x^j$$

Application to a binary pulsar

[Peters, Phys. Rev. 1964]

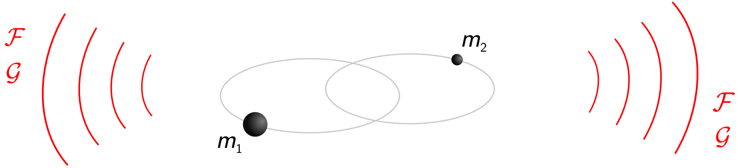


$$\mathcal{F} = \frac{32G^4}{5c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5 (1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right)$$

$$\mathcal{G} = \frac{32G^{7/2}}{5c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)^{1/2}}{a^{7/2} (1 - e^2)^2} \left(1 + \frac{7}{8} e^2\right)$$

Application to a binary pulsar

[Peters, Phys. Rev. 1964]



$$\mathcal{F} = \frac{32G^4}{5c^5}\frac{m_1^2 m_2^2 (m_1+m_2)}{a^5 (1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) = -\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle$$

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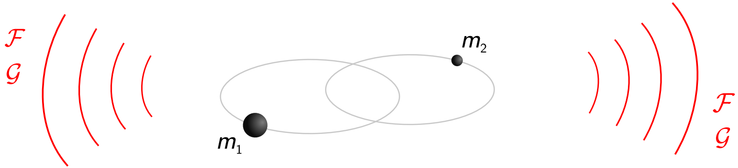
Application to a binary pulsar

[Peters, Phys. Rev. 1964]

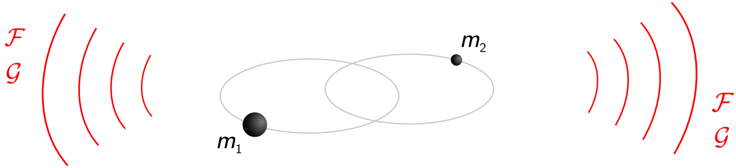


$$\left\langle \frac{\mathrm{d}a}{\mathrm{d}t} \right\rangle = -\frac{64\,G^3}{5c^5}\frac{m_1 m_2(m_1+m_2)}{a^3\,(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = -\frac{304\,G^3}{15c^5}\frac{m_1 m_2(m_1+m_2)}{a^4\,(1-e^2)^{5/2}}\left(e+\frac{121}{304}e^3\right)$$

Application to a binary pulsar

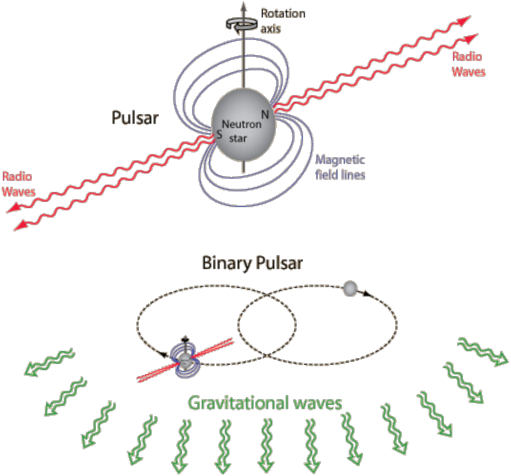
[Peters, Phys. Rev. 1964]



$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}t} \right\rangle = -\frac{192\pi G^{5/3}}{5c^5} \frac{m_1 m_2}{m^{1/3}} \left(\frac{2\pi}{P} \right)^{5/3} \frac{1 + \frac{73}{24} e^2 + \frac{37}{96} e^4}{(1 - e^2)^{7/2}}$$

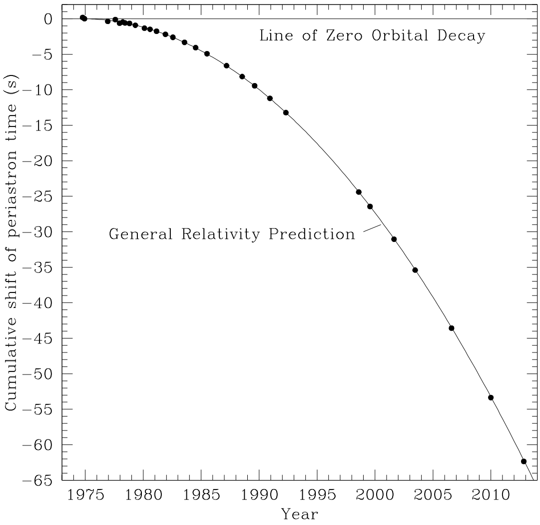
Binary pulsar PSR B1913+16

[Hulse & Taylor, ApJ 1975]



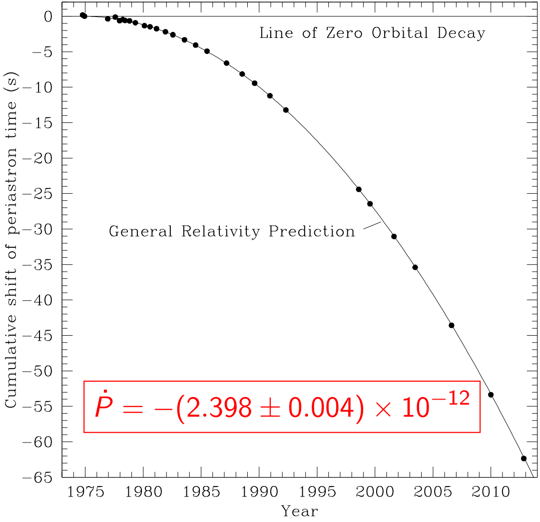
Orbital decay of PSR B1913+16

[Weisberg & Huang, ApJ 2016]



Orbital decay of PSR B1913+16

[Weisberg & Huang, ApJ 2016]



Binary pulsar tests of orbital decay

PSR	$\dot{P}_{int}/\dot{P}_{GR}$	Reference
B1913+16	0.9983 ± 0.0016	Weisberg & Huang (2016)
J0737-3039	1.003 ± 0.014	Kramer *et al.* (2006)
B2127+11C	1.00 ± 0.03	Jacoby *et al.* (2006)
J1756-2251	1.08 ± 0.03	Ferdman *et al.* (2014)
J1906+0746	1.01 ± 0.05	van Leeuwen *et al.* (2015)
J1141-6545	1.04 ± 0.06	Bhat *et al.* (2008)
B1534+12	0.91 ± 0.06	Stairs *et al.* (2001)
J1738+0333	0.94 ± 0.13	Freire *et al.* (2012)
J0348+0432	1.05 ± 0.18	Antoniadis *et al.* (2013)

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- Shapiro delay

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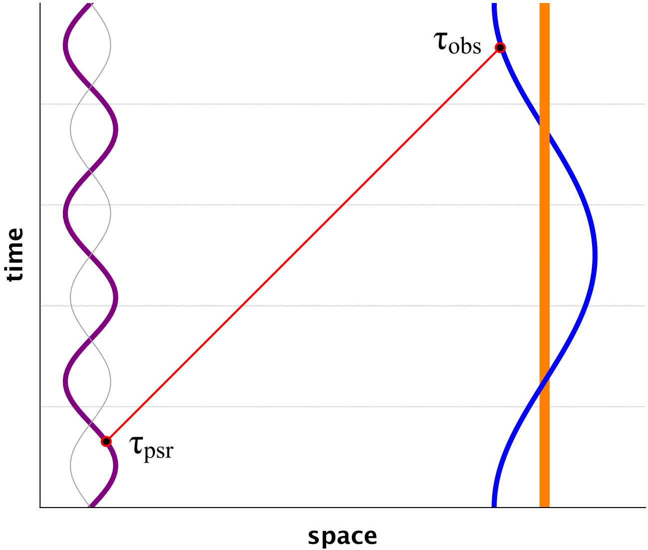
τ_{obs}
τ_{psr}
time
space

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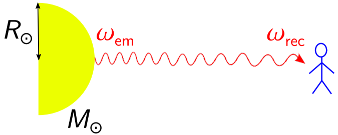
Gravitational redshift of light

- Light frequency

$$\frac{\omega_{\rm em}}{\omega_{\rm rec}} = 1 + \frac{G}{c^2}\frac{M_\odot}{R_\odot}$$



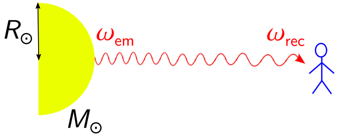
Gravitational redshift of light

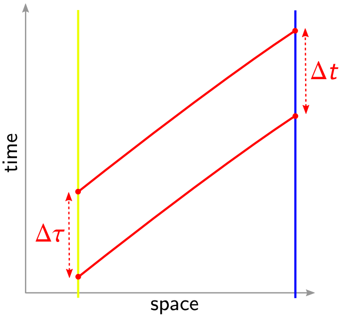
- Light frequency

$$\frac{\omega_{\text{em}}}{\omega_{\text{rec}}} = 1 + \frac{G}{c^2}\frac{M_\odot}{R_\odot}$$

- Proper time

$$\frac{\Delta\tau}{\Delta t} = 1 - \frac{G}{c^2}\frac{M_\odot}{R_\odot}$$





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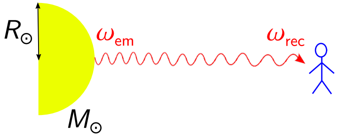
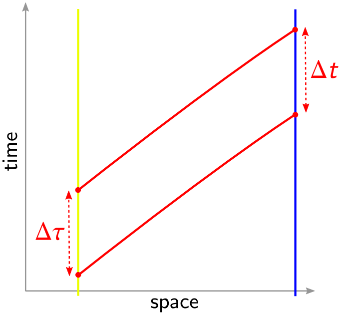
- Confirmed by Gravity Probe A at **0.007%**
  [Vessot *et al.*, PRL 1980]





Einstein delay

$$\frac{\mathrm{d}\tau_{\mathrm{p}}}{\mathrm{d}t} = 1 \overbrace{- \frac{U(\boldsymbol{x}_{\mathrm{p}})}{c^2}}^{\text{gravitat. redshift}} \overbrace{- \frac{1}{2}\frac{v_{\mathrm{p}}^2}{c^2}}^{2^{\text{nd}}\text{ order Doppler}} + \mathcal{O}(c^{-4})$$

Einstein delay

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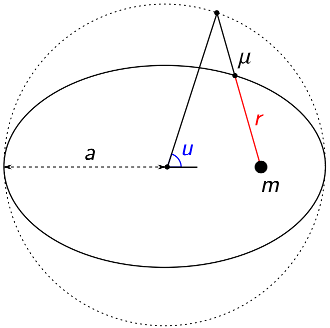
$$= 1 - \frac{Gm_c}{c^2 r} - \frac{m_c}{m}\frac{Gm_c}{c^2 r} + \text{cst.}$$

Einstein delay

$$\frac{\mathrm{d}\tau_p}{\mathrm{d}t} = 1 - \overbrace{\frac{U(\boldsymbol{x}_p)}{c^2}}^{\text{gravitat. redshift}} - \overbrace{\frac{1}{2}\frac{v_p^2}{c^2}}^{\text{2nd order Doppler}} + \mathcal{O}(c^{-4})$$

$$= 1 - \frac{Gm_c}{c^2 r} - \frac{m_c}{m}\frac{Gm_c}{c^2 r} + \text{cst.}$$

$$\downarrow \quad \begin{aligned} r &= a\,(1 - e\cos u) \\ t &= (u - e\sin u)/n \end{aligned}$$



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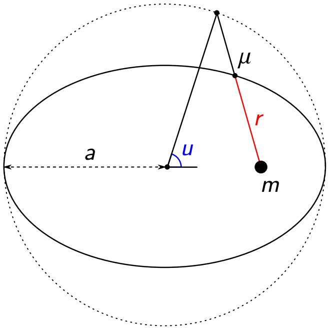
$$\Big\downarrow \quad \begin{aligned} r &= a\,(1 - e\cos u) \\ t &= (u - e\sin u)/n \end{aligned}$$

$$\Delta_\mathrm{E}(u) \equiv \tau_\mathrm{p} - t = \gamma\ \sin u$$



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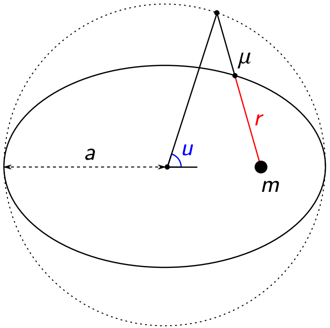
$$\gamma = \frac{Gm_c(m_p + 2m_c)}{c^2 a\,(m_p + m_c)n}\,e$$



Einstein delay

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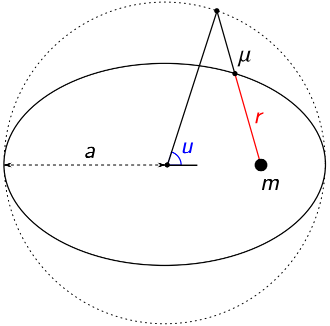

PSR	Amplitude (ms)
B1913+16	4.2992 ± 0.0008
J0737-3039	0.3856 ± 0.0026

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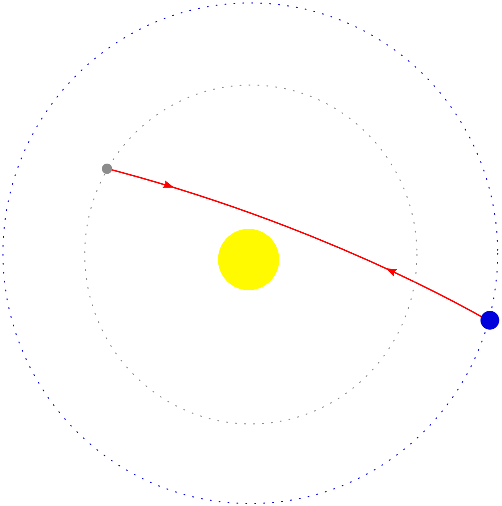
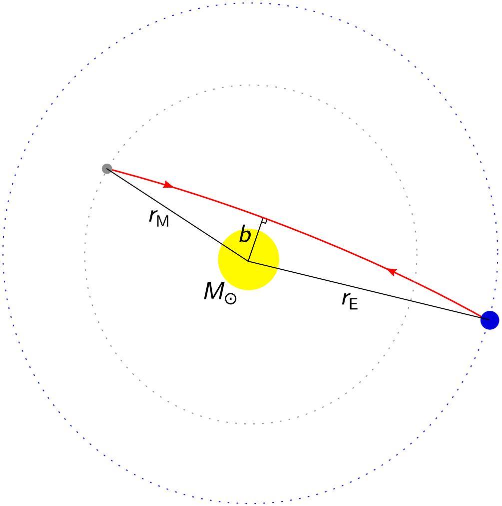
Gravitational time delay

[Shapiro, PRL 1964]



Gravitational time delay

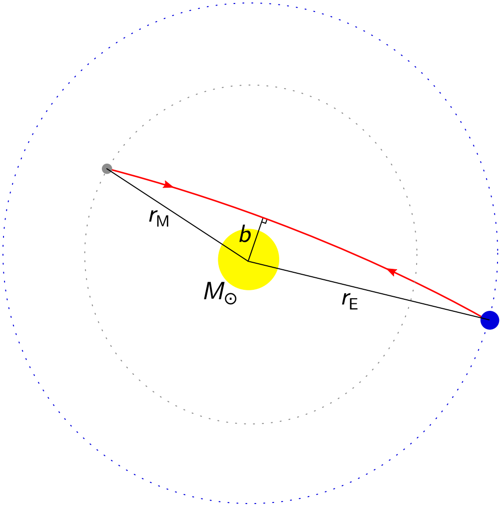
[Shapiro, PRL 1964]



$$\Delta t = \frac{2GM_\odot}{c^3}\left[\ln\left(\frac{4r_E r_M}{b^2}\right)+1\right]$$

Gravitational time delay

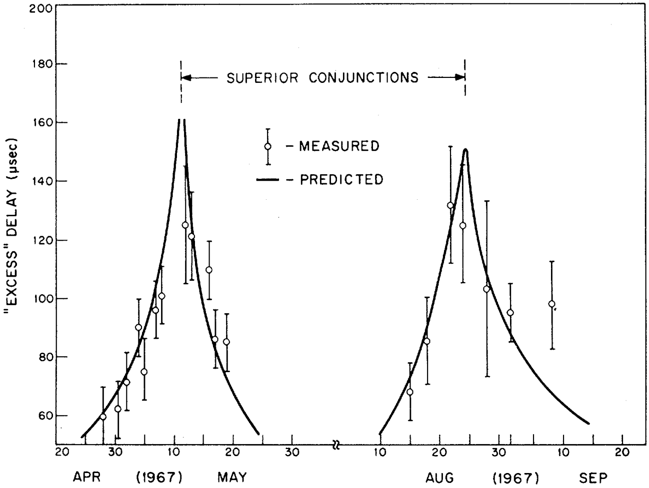
[Shapiro, PRL 1964]



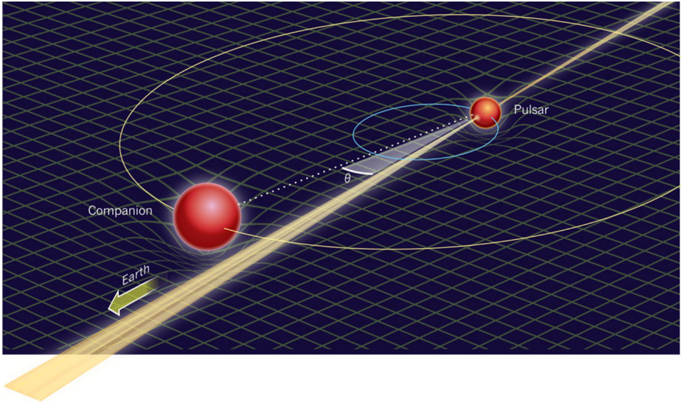
$$\Delta t = \frac{2GM_\odot}{c^3}\left[\ln\left(\frac{4r_E r_M}{b^2}\right)+1\right]$$
$$\simeq 120\ \mu s$$

Gravitational time delay

[Shapiro *et al.*, PRL 1968]



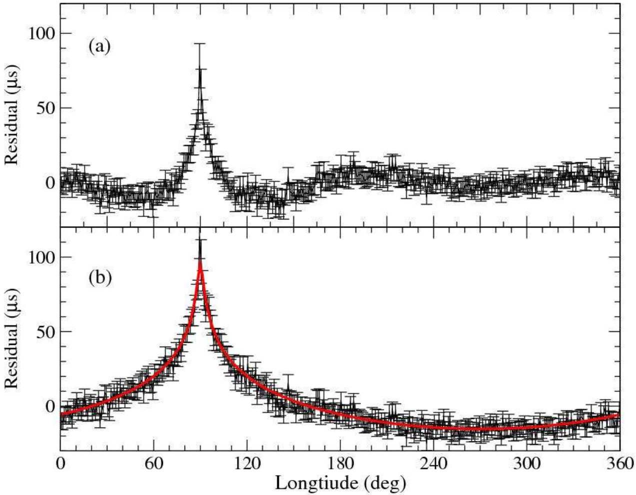
Shapiro delay in binary pulsars



$$\Delta_S(\theta) = -2 \underbrace{\frac{Gm_c}{c^3}}_{\text{range } r} \ln\left(1 - \underbrace{\sin i}_{\text{shape } s} \cos\theta\right)$$

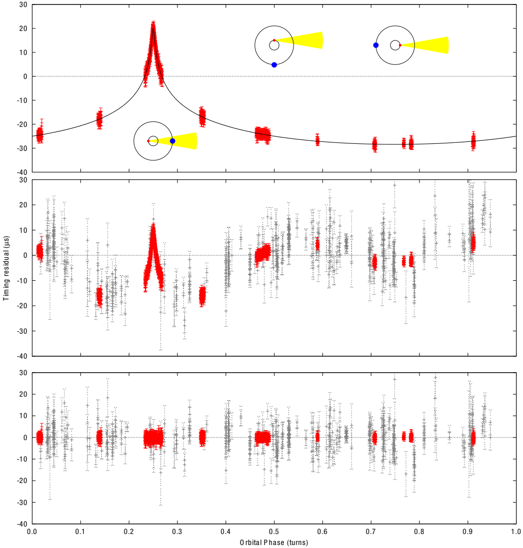
Double pulsar PSR J0737-3039

[Kramer *et al.*, Science 2006]



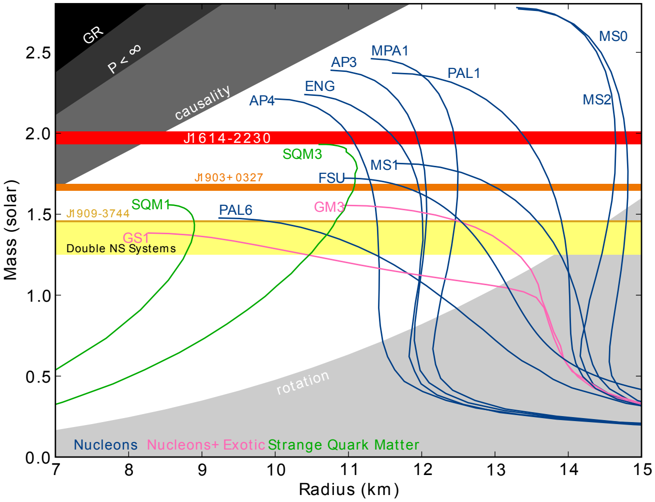
Binary pulsar PSR J1614-2230

[Demorest *et al.*, Nature 2010]



Constraining the NS equation of state

[Demorest *et al.*, Nature 2010]



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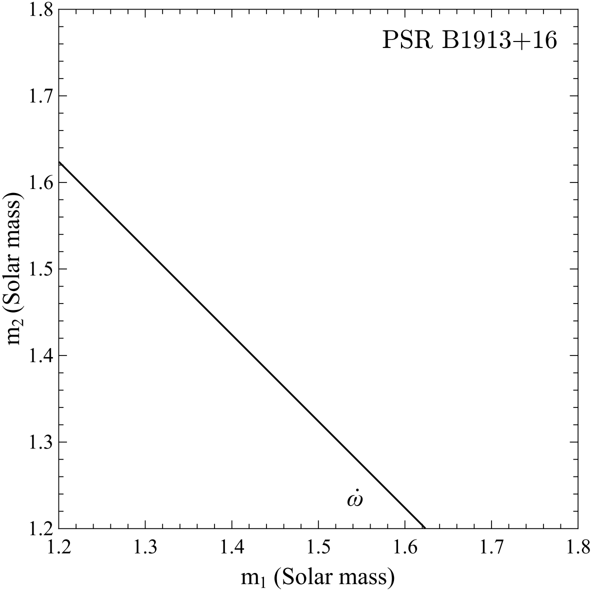
Three post-Keplerian parameters

- Periastron advance

$$\dot{\omega} = \frac{3G^{2/3}}{c^2}\left(\frac{2\pi}{P}\right)^{5/3}\left(1 - e^2\right)^{-1}(m_1 + m_2)^{2/3}$$

Three post-Keplerian parameters



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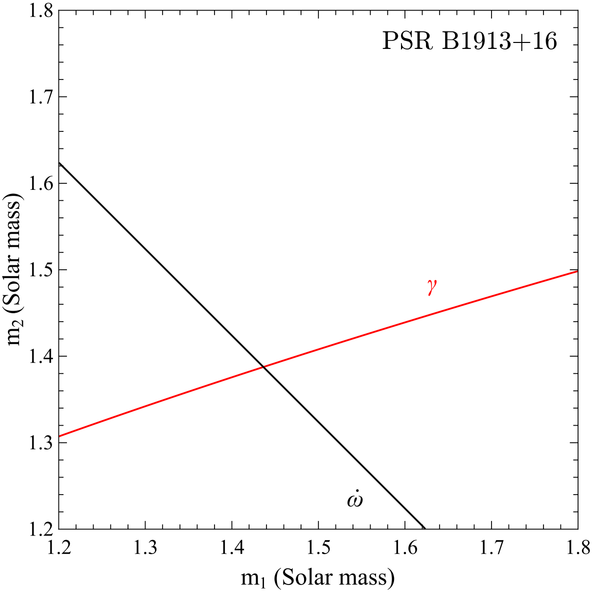
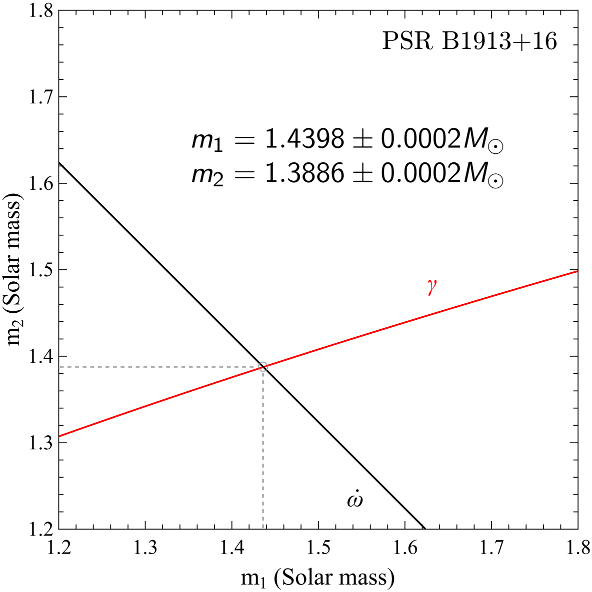
- Einstein delay

$$\gamma = \frac{G^{3/2}}{c^2} \left(\frac{P}{2\pi}\right)^{1/3} e \, \frac{m_2(m_1 + 2m_2)}{(m_1 + m_2)^{4/3}}$$

Three post-Keplerian parameters



Three post-Keplerian parameters



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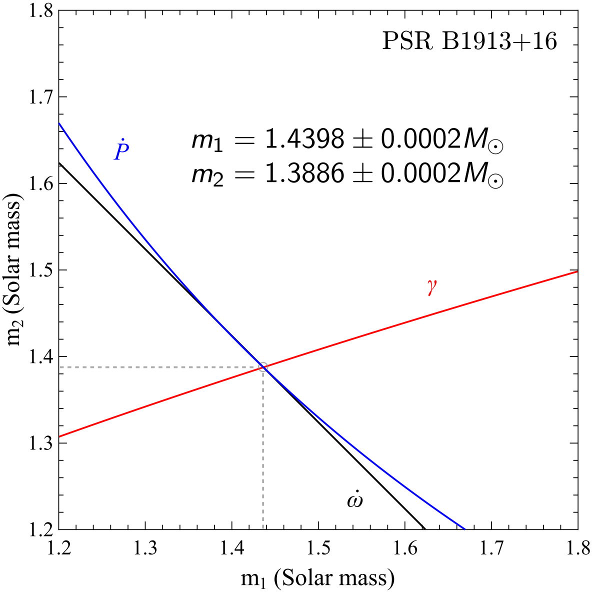
- Orbital decay

$$\dot{P} = -\frac{192\pi G^{5/3}}{5c^5} \left(\frac{2\pi}{P}\right)^{5/3} f(e) \, \frac{m_1 m_2}{(m_1 + m_2)^{1/3}}$$

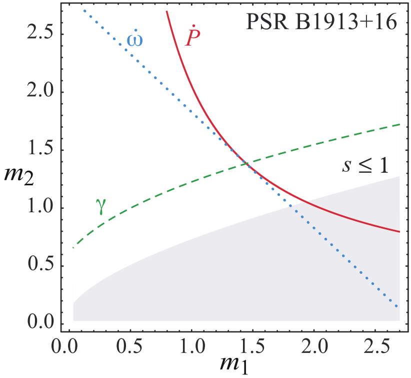
Three post-Keplerian parameters



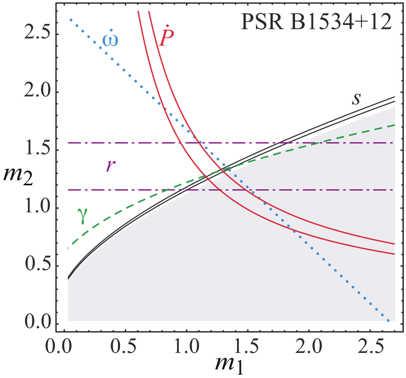
Tests of relativistic gravity

[Esposito-Farèse, Proc. MG10]

1 test



3 tests



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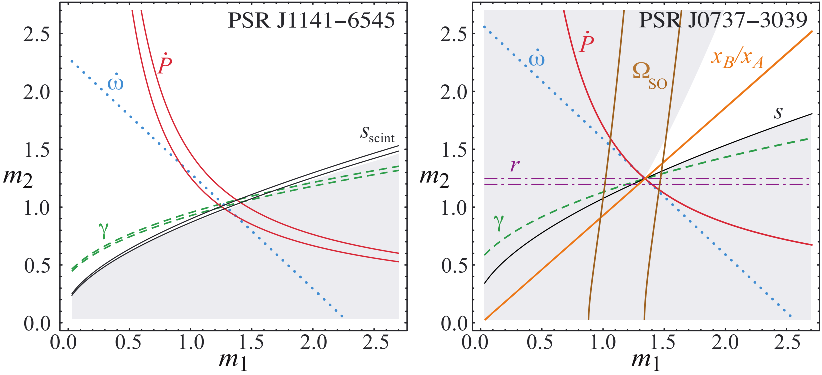
[Esposito-Farèse, Proc. MG10]

2 tests

5 tests



Mass measurements for neutron stars

[Antoniadis *et al.*, ApJ 2016]