Gravitational Radiation

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Outline

The lecture will be divided in different sections:

- Special Relativity (SR).
- General Relativity (GR).
- Gravitational Waves (GWs).
- GW Background (GWB).

References

- Hartle, J.B. (2003) Gravity: An Introduction to Einstein's General Relativity, Benjamin Cummings.

  This is a great and mathematically easy introduction to GR.

- Schutz, B. (2009) A First Course in General Relativity, 2nd edn, Cambridge University Press.

  Also introductory, but a bit more detailed.

- Wald, R.M. (1984) General Relativity, University of Chicago Press.

  Very detailed and mathematical.

- Misner, C.W., Thorne, K.S. and Wheeler, J.A. (1973) Gravitation, Freeman, San Francisco.

  The Bible of Gravity.

- Maggiore, M. (2008) Gravitational Waves Volume 1: Theory and Experiments, Oxford University Press.

  Almost everything you need to know about GWs.

- Creighton, J.D.E., and Anderson, W.G. (2011) Gravitational-Wave Physics and Astronomy: An Introduction to Theory, Experiment and Data Analysis, WILEY-VCH Verlag GmbH & Co. KGaA.

  Also very complete, and more pedagogical, with many interesting exercises.

1 Special Relativity, an overview

Let us consider that SR is a game, with one special rule, which is the principle of relativity: *There is no preferred inertial frame.* When applying this rule to Electrodynamics, one particular consequence is that all inertial frames measure the same speed of light in vacuum, which cannot be exceeded.

Imagine I am running very fast in one direction. How does Nature impede me to travel faster than light? Nature will shrink the distance between me and my final destination, and will make my watch tick slower, so that I can reach my destination without exceeding the speed of light. Those two effects are called space contraction and time dilation. We can derive exactly how these two effects work.

Imagine a train moving on the x direction at constant speed v. Alice is inside the train and Bob is outside. A photon moves upwards from the floor of the train a distance $\frac{1}{2}\Delta y$ and goes back down $\frac{1}{2}\Delta y$, in a time $\Delta\tau$. According to Alice, the photon moves at speed c, so

$$c = \frac{\Delta y}{\Delta\tau}. \tag{1}$$

According to Bob, the photon has traveled, during a time Δt, a distance which is $\sqrt{[\Delta x]^2 + [\Delta y]^2}$, where

$$\Delta x = v\Delta t. \tag{2}$$

To follow the rule of the SR game, the speed of a photon must be the same for both observers, so

$$c = \frac{\sqrt{[\Delta x]^2 + [\Delta y]^2}}{\Delta t}. \tag{3}$$

Using Eqs. (1), (2), and (3), we get

$$\Delta t = \gamma\Delta\tau, \tag{4}$$

where

$$\gamma = \frac{1}{\sqrt{1 - [\frac{v}{c}]^2}}, \tag{5}$$

which is the Lorentz factor, always $\gamma > 1$. We see that time ticks differently for different inertial frames, which is called time dilation.

One can also find simple examples of space contraction. Another interesting consequence of the SR game is that two things may happen at the same time for one observer, and one after the other for a different observer. There are many interesting examples of SR, very often non-intuitive.

Let us define the concept of *event*: it is a point in spacetime, characterised by certain values of the coordinates (t, x, y, z). For convenience, we write these coordinates (x^0, x^1, x^2, x^3), or, in a more compact form, x^α, with $\alpha = 0, 1, 2, 3$. One observer (or inertial frame) may use coordinates x^α, and another observer may use coordinates x'^β. We define the *interval* as

$$[ds]^2 = -c^2[dx^0]^2 + [dx^1]^2 + [dx^2]^2 + [dx^3]^2. \tag{6}$$

This quantity is an *invariant*: it has the same value for all inertial frames. We can check this using the previous example: for Alice, which is the *proper frame* (in which the events happen at the same spatial position),

$$[ds]^2 = -c^2[d\tau]^2, \tag{7}$$

which, using Equations (1) and (3) (and replacing finite intervals Δx by infinitesimal ones dx), is equal to $[ds]^2$ in Bob's frame

$$[ds]^2 = -c^2[dt]^2 + [dx]^2. \tag{8}$$

The interval can be written in a more compact form,

$$[ds]^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu. \tag{9}$$

Using Einstein's summation criterion,

$$[ds]^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{10}$$

In the previous example, as in any other example of SR,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

which is called Minkowski's metric, the metric of a flat spacetime (similar to classical Euclidean space, but with time added as an extra dimension, and the particularity that the speed of light cannot be exceeded). But $g_{\mu\nu}$ can be much more complicated, it can be a function of the coordinates or other parameters. That's what happens in general, in GR.

2 General Relativity, an overview

2.1 Some geometry...

The principle of relativity (already mentioned) says that there is no preferred frame of reference. All physical theories must be invariant under Poincaré transformations, which are the coordinate transformations that relate the x^α with the x'^β within Minkowski's metric.

GR extends this principle to the principle of *general covariance*: *There is no preferred coordinate system at all.* It doesn't matter if you are an inertial frame, or in a roller coaster, doing bungee jumping, or falling into a black hole. Spacetime is a so-called manifold, a 4-dimensional surface, which is measured by different coordinate systems. Given that the metric of spacetime can be anything, sometimes very complicated, we need to "zoom in" in every point of spacetime; if you zoom in enough, the metric will appear flat. "Zooming in" means using infinitesimal intervals of space and time; in other words, using differential geometry.

One can always write the coordinates x'^α as a function of other coordinates x^μ, i.e. there always exist four equations of the form $x'^\alpha = x'^\alpha(x^\mu)$. All physical laws are independent of the choice of coordinates: this is called the *gauge freedom*. All gauge transformations must fulfill

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\alpha} dx'^\alpha. \tag{12}$$

Using Eq. (10),

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} dx'^\alpha dx'^\beta = g'_{\alpha\beta} dx'^\alpha dx'^\beta. \tag{13}$$

Therefore:

$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}. \tag{14}$$

So, given a certain metric and coordinate system, one can always rewrite the metric in different coordinate systems.

At this point GR becomes very mathematically intense... There's always a way to interprete the math in a geometrical sense, but that interpretation requires time, and definitions of different types of derivatives and operations. Let us just define the Christoffel symbols (or connection coefficients) as

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}}. \tag{15}$$

The Riemann tensor is

$$R^{\delta}_{\alpha\beta\gamma} = -\frac{\partial}{\partial x^{\alpha}} \Gamma^{\delta}_{\beta\gamma} + \frac{\partial}{\partial x^{\beta}} \Gamma^{\delta}_{\alpha\gamma} - \Gamma^{\delta}_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma} + \Gamma^{\delta}_{\beta\mu}\Gamma^{\mu}_{\alpha\gamma}. \tag{16}$$

This tensor contains all information about the curvature of spacetime. Two important quantities that are derived from it are the Ricci tensor,

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}, \tag{17}$$

and the Ricci scalar,

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{18}$$

Finally, Einstein's tensor is defined as

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R. \tag{19}$$

One important characteristic of Einstein's tensor is that it has zero divergence,

$$\nabla^{\alpha} G_{\alpha\beta} = 0. \tag{20}$$

All these equations look at first glance extremely complicated (and in general they actually are!). But, as already said, they are just some functions of the metric and the coordinates, and they have a geometrical meaning.

2.2 Some matter...

So far everything was about the geometry of spacetime. Let's now talk about reality: energy and matter. The stress tensor (in classical mechanics) is a 3x3 matrix that gives the infinitesimal force exerted on an element of infinitesimal area of a three dimensional body. The stress-energy tensor combines the energy flux and the stress tensor in a 4x4 matrix, $T^{\alpha\beta}$. Conservation of matter and stress-energy implies that the stress-energy tensor has zero divergence,

$$\nabla^{\alpha} T_{\alpha\beta} = 0. \tag{21}$$

This is also the equation of motion of matter.

The simplest way to relate geometry to matter in a way that the conservation law is respected, is by equating the Einstein tensor to the stress-energy tensor (which both are divergenceless quantities). This is Einstein field equation,

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{22}$$

where the constants are introduced to ensure that we obtain Newton's equations in the limit of slow motion and weak gravitational fields. This equation shows how matter generates the gravitational field.

3 GWs and other solutions to Einstein equation

Einstein equation is very general; one can make up a metric, calculate the Einstein tensor, and the equation will output a possible stress-energy tensor that would be consistent with that metric. However, not every solution is physical, one has to apply some conditions to make the solutions something more than just pure math. A few examples of metrics with physical meaning are:

3.1 Schwarzschild metric

It is the metric outside a spherical mass M (Earth, the Sun, a black hole, etc.), with zero electric charge, zero angular momentum (or less restrictively, a slow rotation), and assuming a zero cosmological constant.

$$ds^2 = -\left[1 - \frac{r_S}{r}\right] c^2 dt^2 + \left[1 - \frac{r_S}{r}\right]^{-1} dr^2 + r^2 d\Omega^2, \tag{23}$$

where

$$r_S = \frac{2GM}{c^2} \tag{24}$$

is the Schwarzschild radius, which is the event horizon of a non-spinning, uncharged black hole. For the Earth, $r_S \approx 9$ mm, for the Sun, $r_S \approx 3$ km.

Generalisations of this metric are:

- Reissner-Nordström metric, for a charged, non-spinning mass.
- Kerr metric, for an uncharged, spinning mass.
- Kerr-Newman metric, for a charged, spinning mass.

3.2 FLRW metric

The Friedmann-Lemaître-Robertson-Walker metric describes an homogeneous, isotropic expanding or shrinking universe.

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega^2\right], \tag{25}$$

where $a(t)$ is the dimensionless cosmological scale factor. This is the metric that describes our Universe at large scales, in the standard model of cosmology.

3.3 Gravitational Waves

We can rewrite Einstein equation in the weak field approximation, where

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{26}$$

where $h_{\alpha\beta}$ is a small perturbation of Minkowski's metric. One obtains the linearized Einstein equation:

$$\Box \overline{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}, \tag{27}$$

Here we have assumed the Lorentz gauge, and the $\overline{h}_{\alpha\beta}$ are the trace-reverse metric,

$$\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h, \tag{28}$$

where h is the trace of $h_{\alpha\beta}$. In vacuum, $T_{\alpha\beta} = 0$, so $\Box \overline{h}_{\alpha\beta} = 0$. One particular solution of this equation is that of a plane wave traveling on the z-direction. Using the transverse-traceless gauge, the obtained metric is

$$ds^2 = -c^2 dt^2 + [1 + h_+(t)]dx^2 + [1 - h_+(t)]dy^2 + 2h_\times(t)dxdy + dz^2, \tag{29}$$

where h_+ and $h_\times$ are the only two independent components, called the plus and cross polarisations of the GW.

The two GW sources that are best modelled are binary systems (for example supermassive black hole binaries) and non axis-symmetric spinning objects (like neutron stars with a certain deformation). The average GW strain amplitude of these sources is:

- For binaries,

$$h = \frac{2[G\mathscr{M}]^{5/3}\pi^{2/3}}{c^4 D} f^{2/3}, \tag{30}$$

where D is the physical distance to the binary, f is the GW frequency, and the *chirp mass* is defined as

$$\mathscr{M} = \frac{[m_1 m_2]^{3/5}}{[m_1 + m_2]^{1/5}}. \tag{31}$$

- For rotating ellipsoids:
$$h = \frac{4\pi^2 G I \varepsilon}{c^4 D} f^2, \tag{32}$$
where I is the moment of inertia about the axis of rotation, and ε is the ellipticity of the ellipsoid.

There are four main types of GW searches:

- Compact binary coalescences (well-modelled binaries in their final stage of inspiral).
- Bursts (unmodelled short signals, like one produced in a supernova).
- Quasi-continuous waves (slow evolving signals, like those produced by spinning neutron stars).
- Stochastic background (a superposition of many weak signals).

For PTAs, the most likely sources of GWs are supermassive black hole binaries. Since they evolve very slowly, their signals are expected to add up, producing a GW background.

4 Gravitational Wave Background

4.1 Some Cosmology...

The cosmological redshift is defined by

$$1 + z = \frac{a(0)}{a(t_e)}, \tag{33}$$

where $a(t_e)$ is the scale factor introduced before in the FLRW metric. The time t_e is conventionally a *look-back time*: it is zero at present and increases as we look back in time. We can rescale the coordinates in such a way that, at present time, $a(0) = 1$. Imagine that we see a supernova, and we determine from its spectrum that it has redshift $z^* = 1$; then, by the time when the explosion happened, t_e^*, the distance to the supernova was half of what its distance now that we observe it, since $a(t_e^*) = 1/2$ and $a(0) = 1$. Equation (33) gives the value of the scale factor at the time of emission of a photon (or a graviton) that is today observed with a redshift z. Differentiating Eq.(33) we obtain

$$\frac{dz}{dt_e} = -\frac{\dot{a}(t_e)}{a^2(t_e)} = -\frac{a(\dot{t_e})}{a(t_e)}\frac{1}{a(t_e)} = -\frac{a(\dot{t_e})}{a(t_e)}[1+z]. \tag{34}$$

Using the definition of the Hubble expansion parameter

$$H(z) = -\frac{\dot{a}(t_e)}{a(t_e)}, \tag{35}$$

we have the relation

$$dt_e = \frac{1}{[1+z]H(z)}dz. \tag{36}$$

The expansion affects intervals of time, frequency and energy of the gravitons (or photons) as follows,

$$df = \frac{df_e}{1+z} \tag{37}$$

$$dE = \frac{dE_e}{1+z} \tag{38}$$

$$dt = dt_e[1+z]. \tag{39}$$

These identities are easy to prove with simple arguments.

The path of the photons (gravitons), moving in a radial direction in space (in a FLRW metric), is obtained by setting $ds^2 = 0$ in Equation (25), which gives

$$dr = \frac{c}{a(t_e)}dt_e = c[1+z]dt_e. \tag{40}$$

Using Eq. (39) we finally get a relation between distances and redshifts,

$$r(z) = \int_0^z \frac{c}{H(z)}dz, \tag{41}$$

which is the comoving distance to the object.

The element of volume of the FLRW (Eq. (25)) metric is

$$dV = a^3(t_e)r^2 d\Omega^2. \tag{42}$$

Considering a uniform distribution of sources, we don't care about angles and integrate over θ and ϕ. Moreover, we prefer to use comoving volumes, $dV_c = a^{-3}(t_e)dV$, so that:

$$dV_c = 4\pi r^2 dr = 4\pi r^2 \frac{c}{H(z)}dz. \tag{43}$$

Since we are assuming that all massive objects are at rest with respect to the cosmological flow (i.e. they move because of the expansion of the Universe, and since comoving coordinates *co-move* with the expansion, those objects appear at rest), no system enters or leaves a certain comoving volume. For this reason it is convenient to measure densities (for example, the number density of systems) per unit comoving volume.

If we impose the FLRW metric and the stress-energy tensor of a perfect fluid (of density ρ and pressure p) in Einstein equation, we obtain Friedmann equation,

$$H^2(t_e) = \frac{8\pi G}{3}\rho(t_e) - \frac{kc^2}{a^2(t_e)} + \frac{\Lambda}{3}. \tag{44}$$

Because we live in a perfect fluid! There's a critical density ρ_c such that, for a zero cosmological constant ($\Lambda = 0$), the curvature k vanishes at present time; that is the density that would keep the Universe closed if there was no dark energy. We can assume a spatially flat universe, $k = 0$, and, using the equation of a perfect fluid for nonrelativistic matter (or dust),

$$\rho_m(z) = \rho_m^0 [1+z]^3, \tag{45}$$

which also comes from Einstein equation in a perfect fluid with FLRW metric. Then, combining Equations (44) and (45) we obtain

$$H(z) = H_0\sqrt{\Omega_m[1+z]^3 + \Omega_\Lambda}, \tag{46}$$

where

$$\Omega_m = \frac{8\pi G\rho_m^0}{3H_0^2} \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2} \tag{47}$$

are the density parameters of matter and dark energy. From observations we get that $(h_0, \Omega_m, \Omega_\Lambda) \approx (0.7, 0.3, 0.7)$, where $H_0 = h_0 100\,\mathrm{km\,s^{-1}\,kpc^{-1}}$ is the current value of the Hubble parameter.

4.2 Some Physics...

We define the GW density parameter (in a similar way as the matter and energy parameter),

$$\Omega(f) = \frac{\rho_{\ln}(f)}{\rho_c} = \frac{\varepsilon_{\ln}(f)}{c^2\rho_c}, \tag{48}$$

so that the total energy density of GWs in the present Universe is

$$\varepsilon_T = \int \varepsilon_{\ln}(f)\, d\ln f. \tag{49}$$

We write the energy density in terms of the energy emitted by one system, dE/df, and the number density of systems, n,

$$\Omega(f) = \frac{1}{c^2\rho_c}\int \frac{dE}{d\ln f} dn = \frac{1}{c^2\rho_c}\int \frac{dE}{d\ln f}\frac{dn}{dt} dt. \tag{50}$$

Using Eqs. (36) and (39),

$$\Omega(f) = \frac{1}{c^2\rho_c}\int \frac{dE}{d\ln f}\frac{dn}{dt}\frac{1}{H(z)}dz. \tag{51}$$

In the quadrupolar approximation,

$$f_e = 2f_{e,\text{orbit}}, \tag{52}$$

Using Kepler's law, the separation between two objects in a binary is given by

$$s = \left[\frac{G[m_1+m_2]}{\pi^2 f_e^2}\right]^{1/3}. \tag{53}$$

In Newtonian mechanics, the binding energy of a binary system is

$$E_{e,sys} = -\frac{1}{2}\frac{Gm_1m_2}{s}. \tag{54}$$

Replacing Equation (53) in (54) and differentiating with respect to f_e, we obtain

$$\frac{dE_{e,sys}}{df_e} = -\frac{1}{3}[G^2\pi^2\mathcal{M}^5]f_e^{-1/3}. \tag{55}$$

Therefore,

$$\frac{dE_e}{d\ln f_e} = -\frac{dE_{e,sys}}{d\ln f_e} = -\frac{dE_{e,sys}}{df_e}f_e = \frac{1}{3}[G^2\pi^2\mathcal{M}^5]f_e^{2/3}. \tag{56}$$

Using Eq. (37),

$$\frac{dE}{d\ln f} = \frac{1}{1+z}\frac{dE_e}{d\ln f_e} = \frac{1}{1+z}\frac{1}{3}[G^2\pi^2\mathcal{M}^5]f_e^{2/3} = [1+z]^{-1/3}\frac{1}{3}[G^2\pi^2\mathcal{M}^5]f^{2/3}. \tag{57}$$

We introduce this in Eq. (51),

$$\Omega(f) = \frac{1}{c^2\rho_c}\int [1+z]^{-1/3}\frac{1}{3}[G^2\pi^2\mathcal{M}^5]f^{2/3}\frac{dn}{dt}\frac{1}{H(z)}dz. \tag{58}$$

Finally, we rearrange and use Eq. (39) again,

$$\Omega(f) = \left[\frac{[G^2\pi^2\mathcal{M}^5]}{3c^2\rho_c}\int [1+z]^{-4/3}\frac{dn}{dt_e}\frac{1}{H(z)}dz\right]f^{2/3}. \tag{59}$$

The characteristic amplitude is related to the density parameter by

$$h_c = \sqrt{\frac{4\rho_c G\Omega(f)}{\pi f^2}}, \tag{60}$$

which we can consider a definition (given that there is no time to prove the relation between strains and energies). So we finally obtain the scaling

$$h_c = A f^{-2/3}, \tag{61}$$

which is often assumed in the PTA literature. In reality the scaling is not exactly that, it actually depends on the model describing the astrophysics of the ensemble of supermassive black hole binaries.