

Gravitational Wave Sources RESULTS

IPTA 2015 Student Workshop

Question 1: h should vary as $1/r$, this way $L \sim r^2 h^2$ is non-zero at infinity.

Question 2: G/c^4 gives you a dimensionless h .

Question 3: G/c^4 is extremely small and GWs are extremely weak. You would need an astrophysical source with large mass and size that vary rapidly with time. A binary system is a good example of that.

Question 4: According to Kepler's third law, the angular orbital frequency Ω is defined by:

$$\Omega^2 = (2\pi/P)^2 = \frac{GM}{a^3} \quad (1)$$

which is true for any eccentricity e .

Question 5: We have:

$$h \sim \frac{1}{r} \frac{\partial^2(MR^2)}{\partial t^2} \sim \frac{1}{r} \Omega^2 \mu a^2 \sim \frac{1}{r} GM \mu / a \quad (2)$$

Adding in the factor G/c^4 we get the needed equation.

Question 6:

$$h \sim \left(\frac{G}{c^4}\right) \frac{1}{r} \mu GM \left(\frac{\Omega^2}{GM}\right)^{1/3} \sim \left(\frac{G}{c^4}\right) \frac{1}{r} \mu \Omega^{2/3} (GM)^{2/3} \quad (3)$$

Question 7:

$$\mu G (GM)^{2/3} = G^{5/3} \mathcal{M}^{5/3} \text{ and } \Omega^{2/3} = \pi^{2/3} f^{2/3} \quad (4)$$

so

$$h \sim \frac{(GM)^{5/3}}{rc^4} (\pi f)^{2/3} \quad (5)$$

Question 8: A rough estimate for the maximum frequency would be when the two black holes are "touching" each other, so that $a = 2R_s$. Then:

$$f_{\max} = \frac{1}{\pi} \left(\frac{GM}{8R_s^3}\right)^{1/2} \quad (6)$$

Question 9: Conservation of energy: Energy at surface of star (kinetic energy, zero gravitational potential) = Energy brought to infinity (zero kinetic energy, gravitational potential): $1/2 mv^2 = GM_{BH}m/r$ so $v_{\text{escape}} = \sqrt{2GM_{BH}/r}$.

Question 10: The escape velocity at the surface of a black hole (the event horizon) is c . so

$c^2 = 2GM_{BH}/R_s$ and $R_s = 2GM_{BH}/c^2$.

Question 11:

$$f_{\max} = \frac{1}{\pi} \left(\frac{c^6}{8G^2M^2} \right)^{1/2} = \frac{c^3}{2\sqrt{2}\pi GM} \quad (7)$$

Question 12: $\mathcal{M}^{5/3} = \mu M^{2/3} = \frac{M}{4} M^{2/3} = \frac{1}{4} M^{5/3}$

$$h = \frac{1}{2} \left(\frac{2}{5} \right)^{1/2} \frac{(GM)^{5/3}}{rc^4} \frac{c^2}{(GM)^{2/3}} = \frac{1}{2} \left(\frac{2}{5} \right)^{1/2} \frac{(GM)}{rc^2} \quad (8)$$

Question 13:

- 1) $h \sim 10^{-21}$, $f_{\max} = 10^3$ Hz
- 2) $h \sim 10^{-18}$, $f_{\max} = 10^{-2}$ Hz
- 3) $h \sim 10^{-14}$, $f_{\max} = 10^{-5}$ Hz

Question 14: PTAs are in some way similar to LIGO, just a much longer interferometer arm (4 km at LIGO, distance Earth - pulsar for PTAs). The PTAs are therefore sensitive to longer wavelengths or shorter frequencies (case 3 for PTAs vs case 1 for LIGO).

Question 15: Flux = $dE/(dAdt)$ is in units of mass/time³. We need a pre-factor of unit mass/time, so the pre-factor is c^3/G .

Question 16: Flux = $\frac{\pi c^3}{4G} f^2 h^2$ and so:

$$dE/dt = \pi^2 r^2 \frac{c^3}{G} \left(\frac{32 (GM)^{10/3}}{5 r^2 c^8} \right) \pi^{4/3} f^{10/3} = \frac{32 \pi^{10/3} G^{7/3}}{5 c^5} (\mathcal{M}f)^{10/3} \quad (9)$$

$$dE/dt = \frac{32 \mu^2 G^{7/3} M^{4/3}}{5 c^5} \pi^{10/3} \frac{1}{\pi^{10/3}} (GM)^{5/3} \frac{1}{a^5} \quad (10)$$

so we have:

$$dE/dt = \frac{32 G^4 \mu^2 M^3}{5 c^5 a^5} \quad (11)$$

Question 17: Equating Equations 23 and 24, we obtain for each frequency (we use $df/f = df_r/f_r$):

$$\frac{\pi c^2}{4G} f^2 h_c^2(f) = \int_0^\infty N(z) \frac{1}{1+z} \frac{dE_{gw}}{df_r} f_r dz \quad (12)$$

therefore:

$$h_c^2(f) = \frac{4G}{\pi c^2} \frac{1}{f^2} \int_0^\infty N(z) \frac{1}{1+z} \left(f_r \frac{dE_{gw}}{df_r} \right) \Big|_{f_r=f(1+z)} dz \quad (13)$$

Question 18: We had previously found (Kepler's Third Law): $\Omega = \sqrt{\frac{GM}{a^3}}$, so that $f_r = \frac{\Omega}{\pi} = \frac{1}{\pi} \sqrt{\frac{GM}{a^3}}$ and:

$$a = \frac{(GM)^{1/3}}{(\pi f_r)^{2/3}} \quad (14)$$

and

$$\frac{da}{df_r} = -\frac{2}{3} \frac{(GM)^{1/3}}{\pi^{2/3}} f_r^{-5/3} \quad (15)$$

Question 19:

$$\frac{dE}{df_r} = \frac{dE}{dt} \left(\frac{1}{da/dt} \right) \frac{da}{df_r} = \left(\frac{32 \pi^{10/3} G^{7/3}}{5 c^5} (\mathcal{M} f_r)^{10/3} \right) \left(-\frac{5}{64} \frac{c^5 a^3}{G^3 \mu M^2} \right) \left(-\frac{2}{3} \frac{(GM)^{1/3}}{\pi^{2/3}} f_r^{-5/3} \right) \quad (16)$$

with:

$$a^3 = \frac{GM}{(\pi f_r)^2} \text{ and } \frac{1}{\mu M^2} = M^{-4/3} \mathcal{M}^{-5/3} \quad (17)$$

so that:

$$\frac{dE}{df_r} = \frac{1}{3} (\pi G)^{2/3} \mathcal{M}^{5/3} f_r^{-1/3} \quad (18)$$

Question 20: Plugging the latest result into Equation 25:

$$h_c^2(f) = \frac{4G}{\pi c^2} \frac{1}{f^2} \int_0^\infty N(z) \frac{1}{1+z} \left(\frac{1}{3} (\pi G)^{2/3} \mathcal{M}^{5/3} f_r^{2/3} \right) dz \quad (19)$$

$$h_c^2(f) = \frac{4(G\mathcal{M})^{5/3}}{3\pi^{1/3} c^2} \frac{1}{f^2} \int_0^\infty N(z) \frac{1}{1+z} \left(f^{2/3} (1+z)^{2/3} \right) dz \quad (20)$$

so that:

$$h_c^2(f) = \frac{4(G\mathcal{M})^{5/3}}{3\pi^{1/3} c^2} f^{-4/3} \int_0^\infty N(z) \frac{1}{(1+z)^{1/3}} dz \quad (21)$$

Therefore:

$$h_c(f) = A f^{-2/3} \quad (22)$$

with:

$$A = \left[\frac{4(G\mathcal{M})^{5/3}}{3\pi^{1/3} c^2} \int_0^\infty N(z) \frac{1}{(1+z)^{1/3}} dz \right]^{1/2} \quad (23)$$