IPTA 2015 Student Workshop
Gravitational Wave Sources

In today's activity, we study the gravitational waves (GWs) produced by binary systems in circular orbits. In Sections 1 and 2, we derive, using dimensional analysis, expressions for the GW amplitude and energy from individual black hole binaries. In Section 3, we use these results to calculate the spectrum of a stochastic background of supermassive black hole (SMBH) binaries, which could be detected by pulsar timing arrays. In general, following the work of Phinney, the spectrum of a GW background by a population of sources can be calculated if we know 1) the total GW energy spectrum of an individual source during its life history (depends on type of source; here we focus on SMBH binaries) 2) an estimate for the number of such sources that have ever lived and died in a unit comoving volume of the universe (and at which redshift).

You will find additional information in the appendices: Section 5 gives a definition of the GW strain amplitude as predicted by General Relativity; Section 6 details the procedure for deriving an exact expression for the GW amplitude from a single binary (same result as Section 2 but with exact coefficients); Section 7 discusses the population of supermassive black hole binaries; and Section 8 mentions the role of the black hole binary environment.

1. Gravitational wave amplitude: orders of magnitude

In this section, we estimate an approximate expression for the GW strain amplitude h, a distance r from a source. There is no monopole or dipole radiation (because of conservation of mass and momentum, respectively). The lowest-order radiation for gravitational radiation is quadrupolar and therefore depends on the mass quadrupole moment $Q_{ij} = \int \rho r_i r_j d^3x$. The quadrupole moment has dimensions MR^2 (M is some mass and R a characteristic dimension). So the GW amplitude h should be proportional to MR^2.

Now how does the GW amplitude decrease with distance from the source? Consider that the flux $F = \frac{dE}{dt\,dA}$ is proportional to the square of the GW amplitude h ($F \sim h^2$), while the luminosity $L = dE/dt$ is such that $L = 4\pi r^2 F \sim r^2 h^2$. If for example the GW amplitude went as $1/r^2$, then we would have: $L \sim 1/r^2$, and the luminosity would be zero at infinity, meaning that the GW would *not* propagate and carry energy to all regions of space.

Question 1: For GW to propagate to infinity, how should the GW amplitude h vary with distance r?

Knowing that GW emission is quadrupolar and knowing how it decreases with distance from the source, we now have:

$$h \sim MR^2/r \tag{1}$$

We know that h is dimensionless, so we don't have the right units, something is missing in the equation. In GR we usually work with "natural units", where $G = c = 1$, meaning that mass, distance and time all have the same "unit". We also know that time derivatives need to be involved, since a static system cannot emit anything. Two time derivatives will cancel out the current units, so we have:

$$h \sim \frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2} \tag{2}$$

To get back to physical units, we have to restore the factors of G and c.

Question 2: Find the correct factors of G and c in the above equation (so that h is dimensionless). Hint: if M is a mass, then GM/c^2 has units of distance, and GM/c^3 has units of time.

Question 3: Since G is small and c is large (the constants are given for reference in the "4: Useful constants" section) , what can you tell about the amplitude of gravitational waves? Which kinds of astrophysical sources might emit the strongest gravitational waves? Why?

2. GW radiation from a binary system in circular orbit: orders of magnitude

In this section, we consider the GW wave emission from a binary system in circular orbit. We want to find expressions for the amplitude and frequency of GW emission, using dimensional analysis. We also want to know the energy lost by the binary, which corresponds to the energy carried away by the GWs. Let's consider binary systems with two masses m_1 and m_2. We can define the total mass $M = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2/M$. A system of two masses m_1 and m_2 in a bound orbit is equivalently described by a single mass μ orbiting in an external potential determined by the total mass of the system M. In a general case, the mass μ moves in an elliptic orbit with eccentricity e and semi-major axis a. Here we assume circular orbits only (so $e = 0$).

Question 4: Using Kepler's third law, find an expression for the angular orbital velocity Ω as a function of the total mass M and the semi-major axis a.

Question 5: Approximating the mass quadrupole moment to be $Q \sim MR^2 \sim \mu a^2$ and replacing $\partial^2/\partial t^2$ with Ω^2 in Equation (2), show that at a distance $r >> a$:

$$h \sim \left(\frac{G}{c^4}\right)\frac{1}{r}\left(\frac{\mu G M}{a}\right) \tag{3}$$

Question 6: Get rid of a in Equation (3) and express the GW amplitude h only as a function of total mass M, r, μ and Ω.

Question 7: We define the chirp mass $\mathcal{M}$ as: $\mathcal{M}^{5/3} = \mu M^{2/3}$. We also know that the frequency of GW emission f is twice the orbital frequency $\Omega/(2\pi)$, i.e. $f = \Omega/\pi$. Starting from the expression you found in the last question, express h as a function of the chirp mass $\mathcal{M}$, r and the GW frequency f.

You have now found expressions for the GW strain amplitude h and the GW frequency f for a binary system in a circular orbit. With the correct constants put in, they are:

$$h = \left(\frac{32}{5}\right)^{1/2} \frac{(G\mathcal{M})^{5/3}}{r\,c^4} (\pi f)^{2/3} \tag{4}$$

$$f = 2 \times \left(\frac{\Omega}{2\pi}\right) = \frac{1}{\pi}\left(\frac{GM}{a^3}\right)^{1/2} \tag{5}$$

Question 8: These two equations are valid for any binary system in a circular orbit. How can we determine the maximum frequency of GW emission for a binary system? In the case of a binary made of two black holes, determine the maximum frequency as a function of the total mass M and the radius of each black hole (called the Schwarzschild radius R_s).

Question 9: Using conservation of energy, find an expression for the escape velocity from an object of mass M_{BH}.

Question 10: The Schwarzschild radius is defined as the point at which light can no longer escape the gravitational pull of the black hole. Using your expression for escape velocity, find an expression for the Schwarzschild radius as a function of the black hole mass M_{BH}.

Question 11: We now assume equal mass black holes so $m_1 = m_2 = M_{BH}$ and M_{BH} can be rewritten as $M/2$. Now express the maximum frequency $f_{\rm max}$ as a function of total mass M.

Question 12: Express the amplitude $h(f_{\rm max})$ as a function of M and r only. With equal mass black holes, $\mu = M/4$ and you can rewrite the chirp mass $\mathcal{M}$ in terms of total mass M.

Question 13 : Using your results from the last two questions, find estimates for the maximum frequency $f_{\rm max}$ and the GW amplitude h (at $f_{\rm max}$) for
1) a binary with two 10-solar mass black holes located at 100 Mpc from Earth
2) a binary with two 10^6-solar mass black holes at 10 Gpc
3) a binary with two 10^9-solar mass black holes at 1 Gpc

Question 14: In which of these scenarios could GWs be detected with pulsar timing arrays (PTAs)? with LIGO? Why are PTAs sensitive to different GW frequencies than LIGO?

Now let us consider flux. We already mentioned that $F \sim h^2$. In fact it's also proportional to the square of the GW frequency, so $F \sim h^2 f^2$.

Question 15: Find the right combination of the constants G and c that will give us the correct unit for flux.

GW luminosity is given by:

$$L = dE/dt = 4\pi r^2 F \sim 4\pi r^2 f^2 h^2 \tag{6}$$

This is the energy lost by the binary as a function of time. The binary loses energy to GWs, which in turn makes the binary moves closer. In the Hulse-Taylor pulsar, a double neutron star system, the binary was seen to be moving closer and losing energy in the exact way predicted by GR, hence providing indirect evidence for the existence of GWs.

Question 16: Show that dE/dt for the binary can be expressed in this way (consider here a general binary with masses m_1 and m_2). In order to get this result, you will need to start from the exact expression for flux: $F = \frac{\pi}{4}\frac{c^3}{G} f^2 h^2$ and use $L = dE/dt = 4\pi r^2 F$.

$$\frac{dE_{gw}}{dt} = \frac{32}{5}\frac{\pi^{10/3} G^{7/3}}{c^5}(\mathcal{M} f)^{10/3} = \frac{32}{5}\frac{G^4}{c^5}\frac{\mu^2 M^3}{a^5} \tag{7}$$

3. Stochastic gravitational wave background

We follow the work of Phinney (see references) who describes the relationship between the spectrum of a GW background, the total-time integrated energy spectrum of an individual source, and the present-day number density of GW sources. This relationship allows us to calculate the amplitude and spectrum of any cosmic gravitational wave background, such as the gravitational wave background of supermassive black hole binaries (SMBHBs), which we hope to detect with PTAs.

We note that so far, we had ignored the expansion of the universe. Indeed, since the universe expands and galaxies appear to be moving away from us, the frequency of the GW radiation coming from these galaxies will be redshifted when observed at Earth. So we need to define two types of frequency: f_r is the GW frequency of the source in its rest frame, while f is the frequency of those waves observed at Earth today. With z being the redshift of the source, we have $f_r = f(1+z)$. The total outgoing energy emitted in gravitational waves between frequency f_r and $f_r + df_r$ is: $\frac{dE}{df_r} df_r$. We define $\Omega_{gw}(f)$ to be the present-day energy density per logarithmic interval in GW of frequency f, divided by the rest-mass energy density $\rho_c c^2$ required to close the universe. The present-day energy density in GW is:

$$\varepsilon_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f) df/f \equiv \int_0^\infty \frac{\pi}{4}\frac{c^2}{G} f^2 h_c^2(f) \frac{df}{f} \tag{8}$$

where $\rho_c = 3H_0^2/(8\pi G)$ is the critical density of the universe, and h_c is the characteristic amplitude of the GW spectrum over a logarithmic frequency interval $dlnf = df/f$ and is the quantity we are interested in. Let us define $N(z)dz$ as the number of GW sources in unit comoving volume between redshift z and $z + dz$. The present-day energy density is equal to the sum of the energy densities radiated at each redshift z, divided by $1/(1+z)$ to account for redshifting:

$$\varepsilon_{gw} \equiv \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} \frac{dE_{gw}}{df_r} f_r \frac{df_r}{f_r} dz \tag{9}$$

Question 17: Show that the characteristic amplitude of GWs is, as a function of frequency:

$$h_c^2(f) = \frac{4G}{\pi c^2} \frac{1}{f^2} \int_0^\infty N(z) \frac{1}{1+z} \left(f_r \frac{dE_{gw}}{df_r} \right) |_{f_r = f(1+z)} dz \tag{10}$$

Now let's apply this equation to the specific case of supermassive black hole binaries. We have found (see Equation 7) (f was in the rest frame therefore f is replaced with f_r):

$$\frac{dE_{gw}}{dt} = \frac{32}{5} \frac{\pi^{10/3} G^{7/3}}{c^5} (\mathcal{M} f_r)^{10/3} \tag{11}$$

We need the following:

$$\frac{dE_{gw}}{df_r} = \frac{dE_{gw}}{dt} \frac{dt}{da} \frac{da}{df_r} \tag{12}$$

Question 18: Differentiating the relation $f_r = \Omega/\pi$, show that:

$$\frac{da}{df_r} = -\frac{2}{3} \left[\frac{GM}{\pi^2} \right]^{1/3} f_r^{-5/3} \tag{13}$$

The term da/dt (a is the semi-major axis as previously defined) represents the orbital decay due to angular momentum losses. From Sesana, Haardt et al (Equation 9), this term behaves as the following when backreaction from GW emission dominates:

$$\frac{da}{dt} = -\frac{64}{5} \frac{G^3}{c^5 a^3} \mu M^2 \tag{14}$$

Question 19: Show that the resulting spectrum of GW is:

$$\left(\frac{dE}{df_r} \right)_{gw} = \frac{(\pi G)^{2/3}}{3} \mathcal{M}^{5/3} f_r^{-1/3} \tag{15}$$

Question 20: Finally, using Equation (15) and Equation (10), show that:

$$h_c(f) = A f^{-2/3} \tag{16}$$

and give an expression for A. This is the behavior we expect for SMBHBs in the later stages of inspiral when GW emission dominates.

4. Useful constants

- $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
- $c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$
- $M_{\odot} = 2.0 \times 10^{30}$ kg
- 1 pc = 3.1 $\times 10^{16}$ m

5. Appendix: Gravitational wave as spacetime perturbation

In Einstein's theory of General Relativity (GR), we characterize the warping of spacetime using the metric tensor $g_{\mu\nu}$. Consider two events determined by four coordinates in a four-dimensional spacetime. If these events are very close to each other, the four-dimensional "distance" ds between these two events is given by:

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{17}$$

where greek indices μ and ν run over all four spacetime coordinates. Here we use the "Einstein summation notation": repeated indices are summed over all coordinates, so that:

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + \text{cross terms such as}\, g_{01} dx^0 dx^1 \text{etc.} \tag{18}$$

The "0" index refers to time, while 1, 2 and 3 represent the usual three-dimensional space. In Cartesian coordinates, these are (c t), x, y and z. Therefore in Cartesian coordinates, the previous equation can be rewritten as:

$$ds^2 = g_{00}\,(c\,dt)^2 + g_{11}\,dx^2 + g_{22}\,dy^2 + g_{33}\,dz^2 + \text{cross terms} \tag{19}$$

The metric tensor, a rank-2 tensor, can be represented as a 4 $\times$ 4 matrix:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{20}$$

In flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

so that

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = -(c\,dt)^2 + dx^2 + dy^2 + dx^2 \tag{22}$$

Note that for a photon traveling at the speed of light, $ds = 0$. When we consider gravitational waves (GWs), we separate the spacetime into a component that is time-independent (the background spacetime) and a component that varies with time (the gravitational waves). If we are far enough from any mass, the background spacetime is flat, and we can write the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu}(\text{flat spacetime}) + h_{\mu\nu}(\text{gravitational waves}), \text{where } h_{\mu\nu} \ll 1. \tag{23}$$

Since $h_{\mu\nu}$ is small, we only need to consider its linear contribution. The Einstein equations describe how matter curves spacetime and how matter moves in curved spacetime. They are however extremely difficult to solve. In the case of a small metric perturbation however, they can then be simplified by the process of "linearization". The linearization of Einstein's equations leads to the wave equation:

$$\Box\overline{h}_{\mu\nu} = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{24}$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - 1/2\,\eta_{\mu\nu}h^{\alpha}_{\alpha}$, h^{α}_{α} is the trace of $h_{\mu\nu}$, and $T_{\mu\nu}$ is the stress-energy tensor representing matter. Far from any source, this reduces to:

$$\Box\overline{h}_{\mu\nu} = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\overline{h}_{\mu\nu} = 0. \tag{25}$$

The solution is a plane wave traveling at the speed of light. As in the case of electromagnetic waves, there are two transverse polarizations. The GW polarizations are represented by the effect they have on a circle of test masses oriented perpendicular to the direction of the wave. These are called the "+" and "×" polarizations. If the GW propagates in the x-direction, the h_+ and $h_\times$ polarizations will vary in the transverse plane, i.e. in the y and z coordinates. We define the "transverse traceless" (TT) gauge as the coordinate system in which the GW polarizations are transverse and the trace of $h_{\mu\nu}$ $(= h_{00} + h_{11} + h_{22} + h_{33})$ is zero. In the TT gauge, The metric $g_{\mu\nu}$ can be written in this way:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 + h^{TT}_{+} & h^{TT}_{\times} \\ 0 & 0 & h^{TT}_{\times} & 1 - h^{TT}_{+} \end{pmatrix} \tag{26}$$

The GW strain h, which is a dimensionless quantity (not to be confused with the trace of $h_{\mu\nu}$ sometimes also called h) represents the ratio of the "stretched part" of the circle to the size of the original circle, or in other words it's the fractional amount by which a circle of test charges is stretched. In this activity, we search for orders of magnitude for the GW strain amplitude, h (whenever we refer to "GW amplitude", we mean the amplitude of the GW strain).

6. Appendix: GW radiation from binary systems in circular orbit: precise derivation

Let's do this again in a more exact way. We consider two masses m_1 and m_2 in a circular orbit, with total mass $M = m_1 + m_2$ and reduced mass $\mu = m_1 m_2/M$. The binary orbit lies on a plane determined by the orbital angular momentum vector $\vec{J}$, while the line of sight (from source to Earth) is defined by the unit vector $\vec{n}$. We define the binary inclination angle i as the angle between the orbital angular momentum vector $\vec{J}$ and the line of sight $\vec{n}$. In this way, an angle of 90^o corresponds to a system that is visible edge-on.

In 1918, Einstein determined that, for GW wavelengths much larger than the source size ($\lambda >> R$), the solution to the linearized Einstein equation (Equation 24) was:

$$h_{\mu\nu} = \frac{2}{r}\frac{G}{c^4}\frac{d^2}{dt^2}\{Q_{\mu\nu}(t - r/c)\}, \tag{27}$$

where $Q_{ij} = \int \rho\, r_i r_j d^3x$ is the mass quadrupole moment in the transverse-traceless (TT) gauge (a simple choice of gauge, i.e. simple choice of coordinate system). This is called the "quadrupole formula" and is an approximation to GR in the newtonian limit (where the stars have orbital velocities $v << c$). The GW amplitude is non-zero only if the quadrupole moment $Q_{\mu\nu}$ and its time derivative vary with time. In a binary in a circular orbit, the mass quadrupole moment is given by the following:

$$Q_{xx} = \bar{Q_{yy}} = \frac{1}{2}\mu a^2 \cos(2\Omega t) \tag{28}$$

$$Q_{xy} = Q_{yx} = \frac{1}{2}\mu a^2 \sin(2\Omega t), \tag{29}$$

where Ω is the orbital angular frequency. From the quadrupole formula (Equation 27) and the quadrupole moments for a binary (Equations 28 and 29) and taking into account the orientation of the binary with respect to the line of sight (angle i), one can find that the GW amplitudes at a distance r from the source are, for each GW polarization (this derivation is a bit too long for the purpose of this worksheet, see Maggiore p. 111 & 158):

$$h_+ = \frac{G^{5/3}}{c^4}\frac{1}{r}\,2\,(1+\cos^2 i)(\Omega M)^{2/3}\mu\,\cos(2\Omega t) \tag{30}$$

$$h_\times = \pm\frac{G^{5/3}}{c^4}\frac{1}{r}4\,\cos i(\Omega M)^{2/3}\mu\,\sin(2\Omega t) \tag{31}$$

The angular frequency of the emitted GWs is therefore $\Omega_{\rm GW} = 2\,\Omega$, that is twice the orbital frequency. The frequency is then $f_{\rm GW} = \Omega_{\rm GW}/(2\pi) = \Omega/\pi$. Let us define the "chirp mass" $\mathcal{M} \equiv \mu^{3/5}M^{2/5}$.

Averaging over the orbital period $P = 2\pi/\Omega$ and the orientations i of the binary orbital plane, we can find an expression for the averaged characteristic GW amplitude $h = \left(< h_+^2 > + < h_\times^2 >\right)^{1/2}$, arriving at Equation 4.

7. Appendix: Supermassive black hole binary population

In order to estimate the amplitude of a background of GWs from SMBH binaries, we need to know how many such sources there are. We start with the assumption that there is a SMBH at the center of each galaxy. This seems to be a reasonably fair assumption since in all cases where the inner part of a galaxy has been resolved, a SMBH was found in the center. Estimating how many SMBH binaries (SMBHBs) could produce GWs is quite complex and depends on:

- The galaxy merger rate (how often do galaxies collide and merge?)
- The relation between SMBHs and their hosts
- The efficiency of SMBH coalescence following galaxy mergers
- when and how accretion is triggered during a merger event

Just because galaxies are seen to be merging does not mean the central black holes will soon merge. It could take a while. The merging of the central black holes happens through *dynamical friction*. Stars in the neighborhood of each black hole navigate through the black hole's gravitational field and gain kinetic energy. In turn, each black hole loses kinetic energy. As the black holes lose energy to neighboring stars, their orbit gets closer. When they are separated by a fraction of a parsec, gravitational radiation takes over and the black holes continue to spiral toward each other. Now whether the black hole is immersed in a dominantly stellar or gaseous environment affects the point at which gravitational radiation takes over (see next section).

8. Appendix: Black hole binary environment and GW emission

In a galaxy merger, the central SMBHs are initially inspiralling toward each other because of dynamical friction. The BHs are coupled to their environment, be it stellar-driven or gas-driven. This causes the BHs to lose energy and to get closer. Only when the BHs get within parsecs of each other does GW emission in the PTA range start to occur. The orbit of the binary is assumed to have a certain eccentricity during the initial inspiral (which is driven by dynamical friction). However, once GW emission takes over and the black holes "decouple" from their environment, the GW emission process is thought to "circularize" the orbit. We therefore usually assume circular orbits when trying to detect GWs with PTAs. However, this may not be true. If the SMBHB stays coupled to its environment until later stages, the orbit could still have great eccentricity as the GW emission process starts. The eccentricity of the SMBH binary would in turn greatly affect the expected GW emission signal. The angular separation at which GWs start to dominate depends on whether the BH binary is in a stellar or gaseous environment. We follow the work of Roeding & Sesana. Defining the GW energy loss timescale as:

$$t_{GW} = 7.84 \times 10^7 \mathrm{yr}\, M_8 q_s^{-1} a_3^4 F(e)^{-1}, F(e) = (1-e^2)^{-7/2}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \tag{32}$$

where $q_s = 4q/(1+q^2)$ is the symmetric binary mass ratio, $M_8 = M/10^8 M_\odot$, and a_3 is the binary separation in units of $10^3 R_s$ with R_s the Schwarzschild radius ($R_s = 2GM/c^2$). In a gas-rich environment (assuming a β-disc model), the migration timescale is:

$$t_m = 2.09 \times 10^6 \mathrm{yr}\, \alpha_{0.3}^{-1/2} \left(\frac{\dot{m}_{0.3}}{\epsilon_{0.1}}\right)^{5/8} M_8^{3/4} q_s^{3/8} \delta_a(e)^{7/8} a_3^{7/8}, \tag{33}$$

where $\alpha = \alpha/0.3$ is the viscosity parameter, $\epsilon_{0.1} = \epsilon/0.1$ is the accretion efficiency and $\dot{m} = \dot{m}/0.3$ is the mass accretion rate with $\dot{m} = \dot{M}/\dot{M}_{Edd}$ the accretion rate normalized to the Eddington rate. In a star-dominated system, the MBHB hardening timescale is given by:

$$t_h = 2.89 \times 10^6 \mathrm{yr}\, \sigma_{100} M_8^{-1} \rho_5^{-1} a_3^{-1} H_{15}^{-1} \tag{34}$$

Equating the migration timescale to the GW timescale, one obtains the angular separation a at which the GW emission takes over in the binary evolution for a gas-rich environment. Equating the hardening timescale to the GW timescale, one obtains the angular separation at which the GW emission takes over for a star-dominated environment. We find the SMBHBs are still coupled to their environment down to a separation of a few hundred Schwarzschild radii. This means that SMBHBs are generally eccentric, even close to coalescence. This strongly affects our assumption that the orbits are circular. Hot topic right now. Stay tuned!

9. References

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