Detecting Gravitational Waves with a pulsar timing array

Key Concept in General Relativity: Mass curves space-time

Picture of curved space time

- First observed during solar eclipse
- Light from stars behind the sun was visible due to curvature

Picture of lensing

More extreme example:
Lensing due to massive foreground galaxy

Observed in binary pulsars

- Shapiro delay due to companion

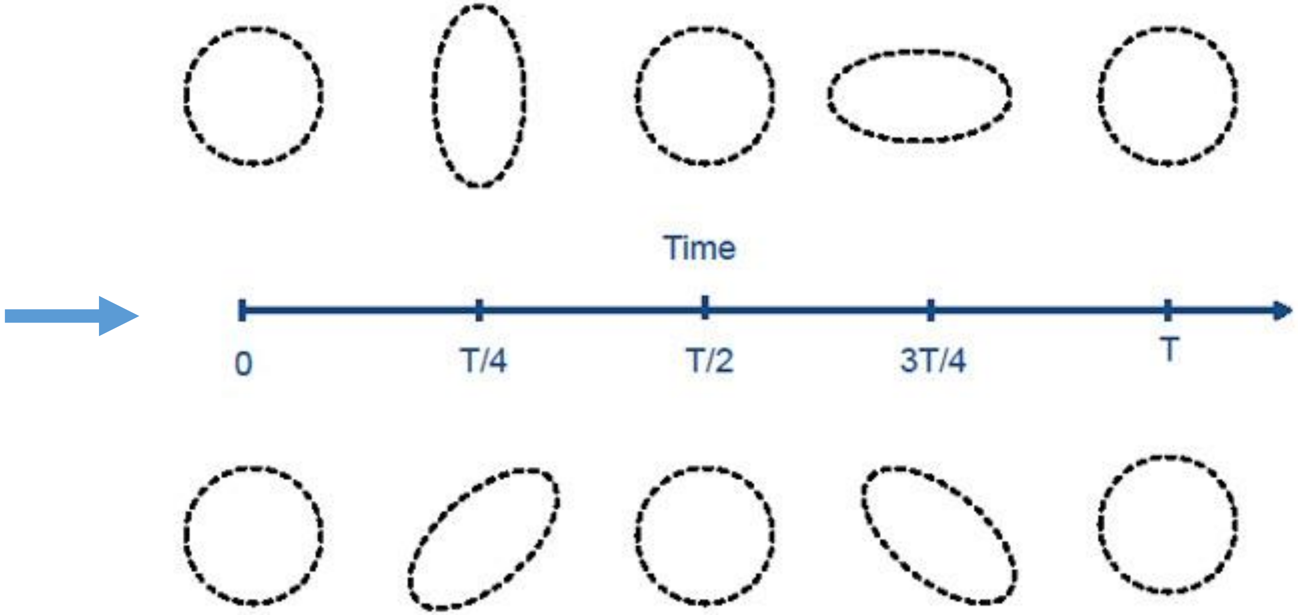
Picture of Shapiro delay effect

Picture of what it looks like in residuals

Gravitational Waves

- Predicted by General Relativity, but have yet to been directly detected.

Experiment in the lab:



- Using an accurate clock, record when you measure ticks.
- Passing gravitational waves contract/expand space between observer and the clock.
- Measure deviation from expected time -> Detected gravitational waves.
    - Expected length change: width of an atom in the Earth-Sun distance

Small effect described as a perturbation of flat space

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \longrightarrow \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \quad + h_{ij}$$

Metric perturbation due to monochromatic source has plane wave expansion:

$$h_{ij}(t, f, \Omega) = \sum_A \exp\left(2\pi i f(t \ - x\Omega)\right) h_A(f, \Omega) e_{ij}^A(\Omega)$$

Have a perturbation due to GW at the pulsar at a time t_p, and at the Earth at a time t_e, so we measure the difference between the two:

$$\Delta h_{ij} = h_{ij}\left(t_p, \Omega\right) - h_{ij}(t_e, \Omega)$$

Assume amplitude of the perturbation is same at both points:

$$\Delta h_{ij} = \sum_A (\exp(2\pi i f t \ - 2\pi i f L(1 + \Omega\hat{p}))\, h_A(f, \Omega) e_{ij}^A(\Omega)$$

$$- \exp(2\pi i f t)\, h_A(f, \Omega) e_{ij}^A(\Omega))$$

$$= \exp(2\pi i f t)\ (\exp(-2\pi i f L(1 + \Omega\hat{p}) - 1) \sum_A \ h_A(f, \Omega) e_{ij}^A(\Omega)$$

For a pulsar with period ν_o the change in path length results in a change in the observed period, or a redshift

$$z(t,\Omega) \equiv \frac{\nu_o - \nu_p}{\nu_o} = \frac{1}{2}\frac{p^i p^j}{1+\Omega p}\Delta h_{ij}$$

Choose Coordinates:

$$t_p = t_e - L \equiv t - L$$
$$x_e = 0$$
$$x_p = L\hat{p}$$

Total red shift obtained by integrating over all frequencies and all sky

$$z(t) = \int df \int_{S^2} d\Omega\, z(t, f, \Omega)$$

Don’t observe redshift, observe a fluctuation in the residuals left after subtracting a timing model from data

$$\mathrm{R}(t) = \int_0^t dt' z(t')$$

Eg:

Picture of residuals from MDC 1.0 showing GWs

Real pulsars have possible sources of noise other than GWs, instrinsic red, DM etc

Picture of 1939 residuals

Need some way to distinguish between intrinsic noise, and fluctuations due to a gravitational wave background

Solution:

Signal due to GWs is correlated between pulsars, other sources of noise (mostly) are not.

Assumptions about background:

Radiation from different directions , statistically independent.
Polarizations , statistically independent
Frequencies , statistically independent.

So, expectation value of amplitude:

$$< h_A^*(f,\Omega)h_{A'}(f',\Omega') >= \delta^2(\Omega,\Omega')\delta_{AA'}\delta(f\ -f')H(f)P(\Omega)$$

Seperated power spectrum so that $P(f,\Omega) =\ H(f)P(\Omega)$, a spectral and spatial part.

Need to work out correlation between residual from pulsar a at time t_j with a pulsar b at a time t_k

Ie:

$$< r_a^*(t_j) r_b(t_k) > = < \int^{t_j} dt' \int^{t\_k} dt'' z_a^*(t') z_b(t'') >$$

$$= < \int^{t_j} dt' \int^{t\_k} dt'' z_a^*(t') z_b(t'') >$$

$$= \int^{t_j} dt' \int^{t_k} dt'' \int_{-\infty}^{\infty} df \ \exp(-2\pi i f(t' - t'') H(f) \, {}^{ab}\Gamma(\mathrm{f})$$

$${}^{ab}\Gamma(f) = \int d\Omega P(\Omega)(\exp(-2\pi i f L_a(1 + \Omega \hat{p}_a)) - 1)(\exp(-2\pi i f L_b(1 + \Omega \hat{p}_b)) - 1) \times \sum_A F_a^A(\Omega) F_b^A(\Omega)$$

For an isotropic background $P(\Omega)$ is a constant for all sky positions

In addition fL will be large, 10-1000

Plot of ORF as a function of fL from anholm

For relevant values of fL the frequency dependent pulsar term part goes to zero, can approximate with:

$$^{\mathrm{ab}}\Gamma(f) = P\int \mathrm{d}\Omega \sum_{A} F_a^A(\Omega)F_b^A(\Omega)$$

Gives Hellings Downs Curve Normalisation so that ORF is 1 for separation of 0 :

$$^{ab}\Gamma = 3\left(\frac{1}{3} + \frac{1 - \ cos\epsilon}{2}\left[\ln\left(\frac{1\ -\ cos\epsilon}{2}\right) - \frac{1}{6}\right]\right)$$

Plot of HD Curve

Smoking Gun of a real GW detection.

Plot of reconstruction of HD Curve from MDC1

So how do we do this for real

P(model given data) = P(data given model)*P(model)

Need to build up the model

Start with a single pulsar

Simplest model:

Data = time of arrivals

Model = deterministic timing model

e.g. J0030 fit position, spin down, proper motion, parallax and DM (variations induced due to ISM)

Plots of some of these basis functions

Start with a single pulsar

P(arrival times | timing model (ϵ))

= $\exp\left(\left(d \ - \tau(\epsilon)\right)^T N^{-1}\left(d \ - \tau(\epsilon)\right)\right)$

$N_{ii} = \sigma_i^2$ = uncorrelated white noise associated with arrival time i

P(timing model) = 1 (Uniform Prior)

Dimensionality of timing model ~6-50 parameters depending on pulsar or PTA.

Start with a single pulsar

This assumes that there is only white noise (unlikely) and that we know its value (unlikely).

Start with a single pulsar

Modify white noise using multipliers (EFAC) that represent uncertainty in the thermal (instrumental) noise.

Add a term in quadrature (EQUAD) that models high frequency noise intrinsic to pulsar

$$\hat{\sigma}_i^2 = (EFAC * \sigma_i)^2 + EQUAD^2$$

Have to do this for every observing ‘system’ (ie, telescope + backend combo)

Anywhere from 1 to 40 systems per pulsar -> 2 to 80 white noise dimensions

Start with a single pulsar

P(d | ϵ, efacs, equads)

$$= \exp\left(-0.5 * \left(d \; - \tau(\epsilon)\right)^T \widehat{N}^{-1}\left(d \; - \tau(\epsilon)\right)\right)$$

Now need red noise, long term variations in the arrival times

(as seen in 1939)

Start with a single pulsar

P(d | ϵ, efacs, equads, a)

= $\exp\left(-0.5(d\ -\tau(\epsilon)\ -Fa)^T\hat{N}^{-1}(d\ -\tau(\epsilon)\ -Fa)\right)$

a's are Fourier Coefficients, describe signal in time domain.

Prior on a's:

P(a | $\phi) = \exp(-0.5a^T\Psi^{-1}a)$

Start with a single pulsar

P(a | Ψ) $= \exp(-0.5 a^T \Psi^{-1} a)$

Gaussian prior on the a's, either model unparameterised power spectrum so that $\Psi_{\mathrm{ii}} = 10^{-\phi_i}$

Or ϕ represent a power law or some other model

Eg: plots of power spectrum of 1939 with time domain signal

Start with a single pulsar

We handle Dispersion measure variations in a similar way.

Basis vectors just have a scaling of 1/(observing frequency)^2

Problem:

Done like this dimensionality is *huge*: worst case

50 timing model parameters

80 white noise parameters

100 a’s for both red noise and DM (including periods up to 50/T)

Do some Bayesian trickery:

Marginalise over some of these parameters analytically.

(are you going to cover marginalisation? If not:)

Want to account for our uncertainty in timing model parameters, and the fourier coefficients, without actually sampling them directly.

Equation of marginalisation

Do some Bayesian trickery:

Plots of 2d example doing marginalisation numerically

But can also solve this equation analytically (could do this on the board)

Do some Bayesian trickery:

Final equation for single pulsar analysis after marginalising over all the things here.

Now ‘only’ ~80 dimensions for a single pulsar.

Downside: introduces dense matrix inversions (computationally costly)

Full PTA Analysis (Searching for GWB):

Have the same set of parameters for each pulsar in the pta (~40)

Each pulsar has its own covariance matrix describing the noise in that pulsar, build up total covariance matrix:

Figure of block diagonal matrix with individual sigmas

Full PTA Analysis:

Then add GWB: introduces cross terms, correlation described by Hellings-Downs curve:

Figure with added cross terms

Full PTA Analysis:

Still not done

Have to consider other sources of correlation:

Clock errors

Errors in planetary ephemeris

Uncorrelated but common timing noise

Full PTA Analysis:

Dimensionality ~250

Challenging to sample

Might take a month to get enough independent points.

Full PTA Analysis:

Other stuff?