An introduction to General Relativity and Gravitational Waves

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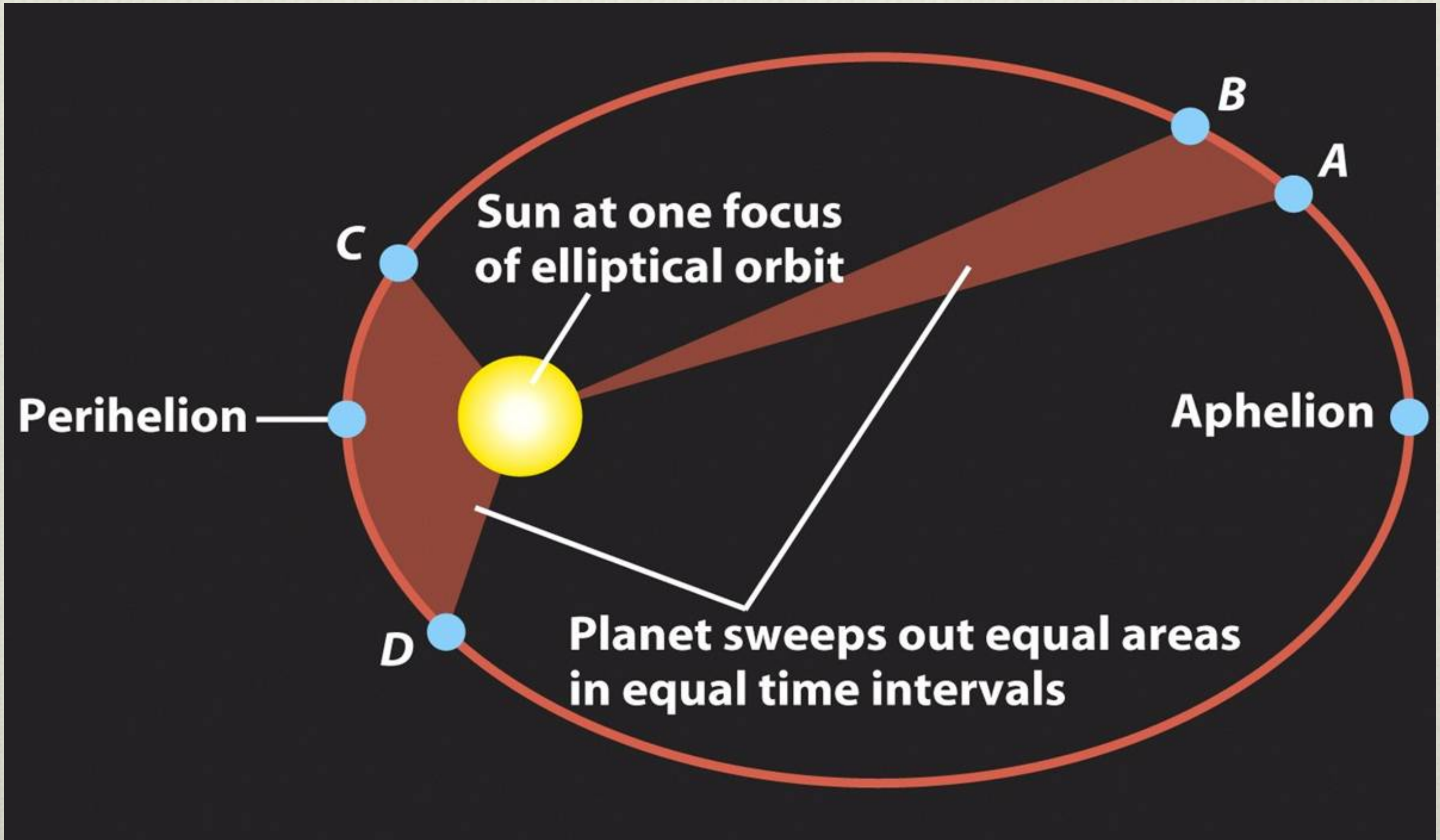
Outline

- Newtonian Gravity
- Problems with Galilean relativity
- Understanding general relativity
- Predictions of general relativity
- Gravitational Waves

Historical context

- What is the preferred approach to compute the orbits of celestial objects?
- Is the true motion of celestial objects complicated and irregular?



Historical context

- Is there some empirical basis to resolve the controversy between the Copernican versus the Tychonic system?

Newton’s Principia

- Newton defined absolute time, space and motion in his Principia as follows:

  - Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration

  - Absolute space, in its own nature, without regard to anything external, remains always similar and immovable

  - Absolute motion is the translation of a body from one absolute place into another

Newton's Principia

- Newton aimed to describe how a specific quantity can be measured
- Introduced terms such as mass, inertia and centripetal force that remained a part of physics ever since
- Caveats: *It is certainly very difficult to find out the true motions of individual bodies and actually to differentiate them from apparent motions, because the parts of that immovable space in which the bodies truly move make no impression on the senses. But the situation is not utterly hopeless…*

Newton's Principia

- Solution: *A fuller explanation will be given of how to determine true motions from their causes, effects, and apparent differences, and, conversely, of how to determine from motions, whether true or apparent, their causes and effects. For this was the purpose for which I composed the following treatise…*

- A solitary rotating sphere can be inferred to rotate about its axis relative to absolute space by observing the bulging of its equator

- A solitary pair of spheres tied by a rope can be inferred to be in absolute rotation about their center of gravity by observing the tension in the rope

Introducing Gravity

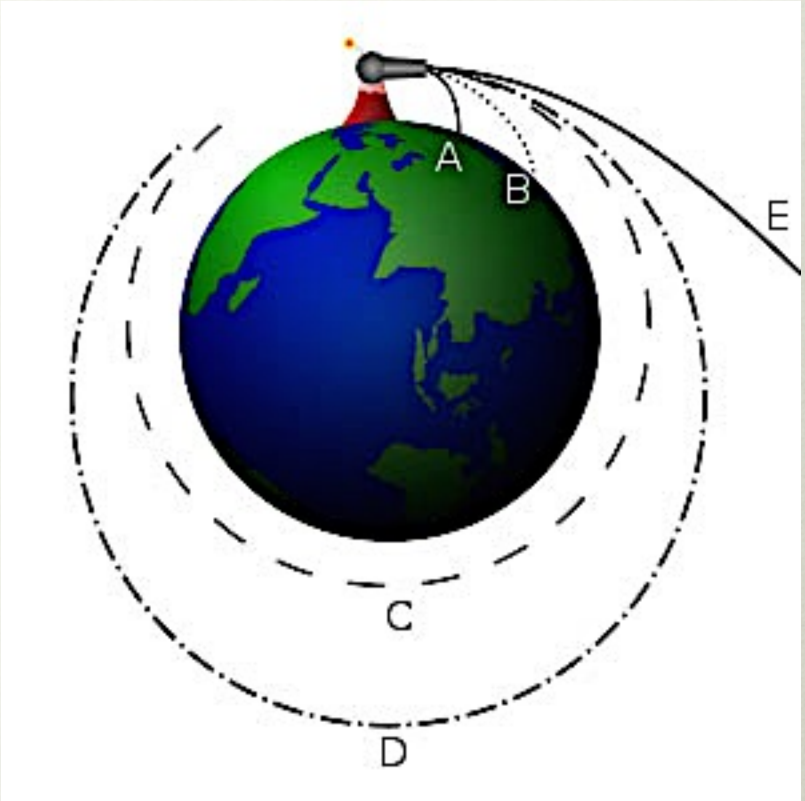
"I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called a hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction."

Universal Gravitation

Using his rules of Reasoning in Philosophy, Newton deduced that:

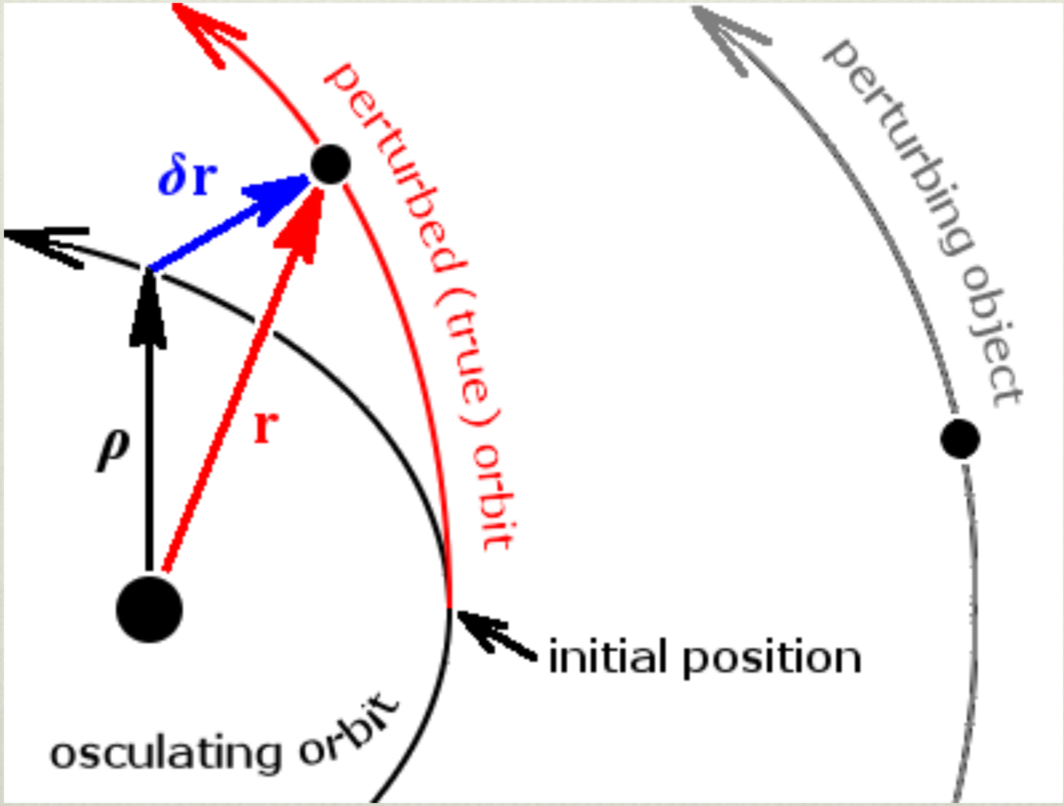
The force of gravity associated with each planet goes as the inverse square of the distance from the planet's center.

There is a power of gravity pertaining to all bodies, proportional to the several quantities of matter they contain



Universal Gravitation

All the planets gravitate towards one another and can influence their orbits



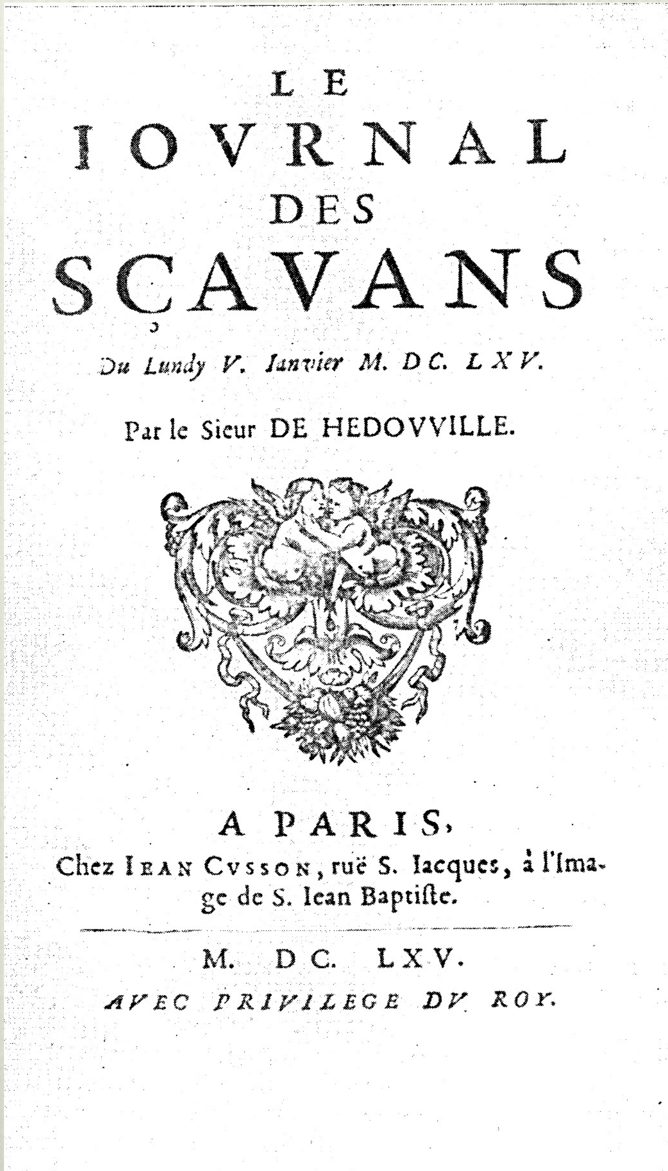
Critical response to the Principia

Christiaan Huygens:

I am not especially in agreement […] that all the small parts that we can imagine in two or more different bodies attract one another or tend to approach each other mutually. This I could not concede, because I believe I see clearly that the cause of such an attraction is not explicable either by any principle of Mechanics or by the laws of motion

Critical response to the Principia

Journal des Sçavants: The work of M. Newton is a mechanics, the most perfect that one could imagine… But one has to confess that one cannot regard these demonstrations otherwise than as only mechanical; indeed the author recognizes that he has not considered their Principles as a Physicist, but as a mere Geometer…



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DES
SÇAVANS
Du Lundy V. Ianvier M. DC. LXV.
Par le Sieur DE HEDOVVILLE.

A PARIS,
Chez IEAN CVSSON, ruë S. Iacques, à l'Image de S. Iean Baptiste.

M. DC. LXV.
AVEC PRIVILEGE DV ROY.

Newtonian Gravity

- It has a preferred class of observers: inertial (non-accelerating) observers

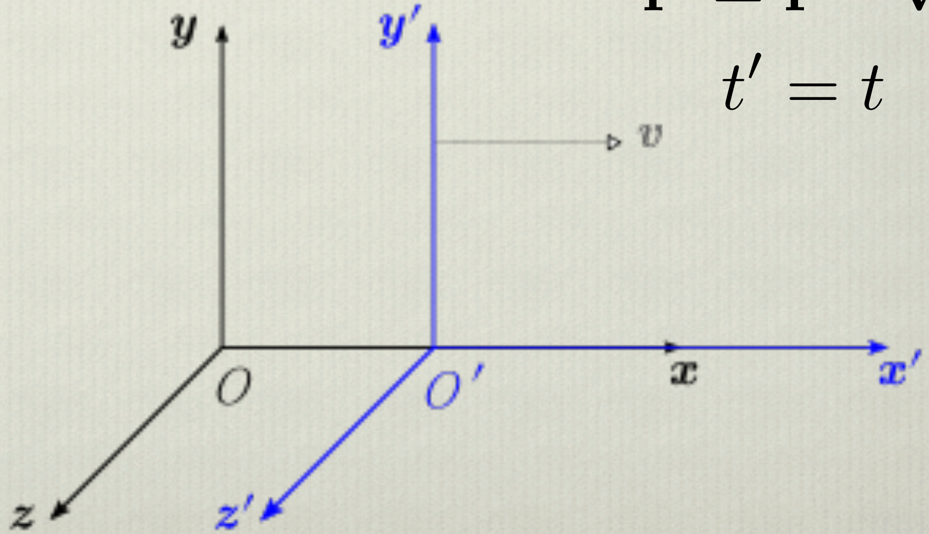
- Any observer has an associated set of coordinates (t, x, y, z) called an inertial frame

- Inertial frames are related by Galilean transformations:

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$$

$$t' = t$$

The geometry of space is specified in Cartesian coordinates by the line element:
$\mathrm{d}S^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$



Newtonian Gravity

Newton's law of gravity

$$\Phi(t,\,\mathbf{x}) = -G\int \mathrm{d}^3\mathbf{y}\,\frac{\rho(t,\,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

Indicates that the gravitational field $\Phi(t,\,\mathbf{x})$ responds instantaneously to changes in the mass density $\rho(t,\,\mathbf{y})$

Approximation valid when $|\Phi|/c^2 \ll 1$

In the Solar System: $|\Phi|/c^2 < 10^{-5}$

What about invariance

- Dynamical equations in Newtonian mechanics preserve their form under Galilean transformations

- Consider a coordinate transformation consisting of a rotation in the xy-plane by an angle θ about the z-axis

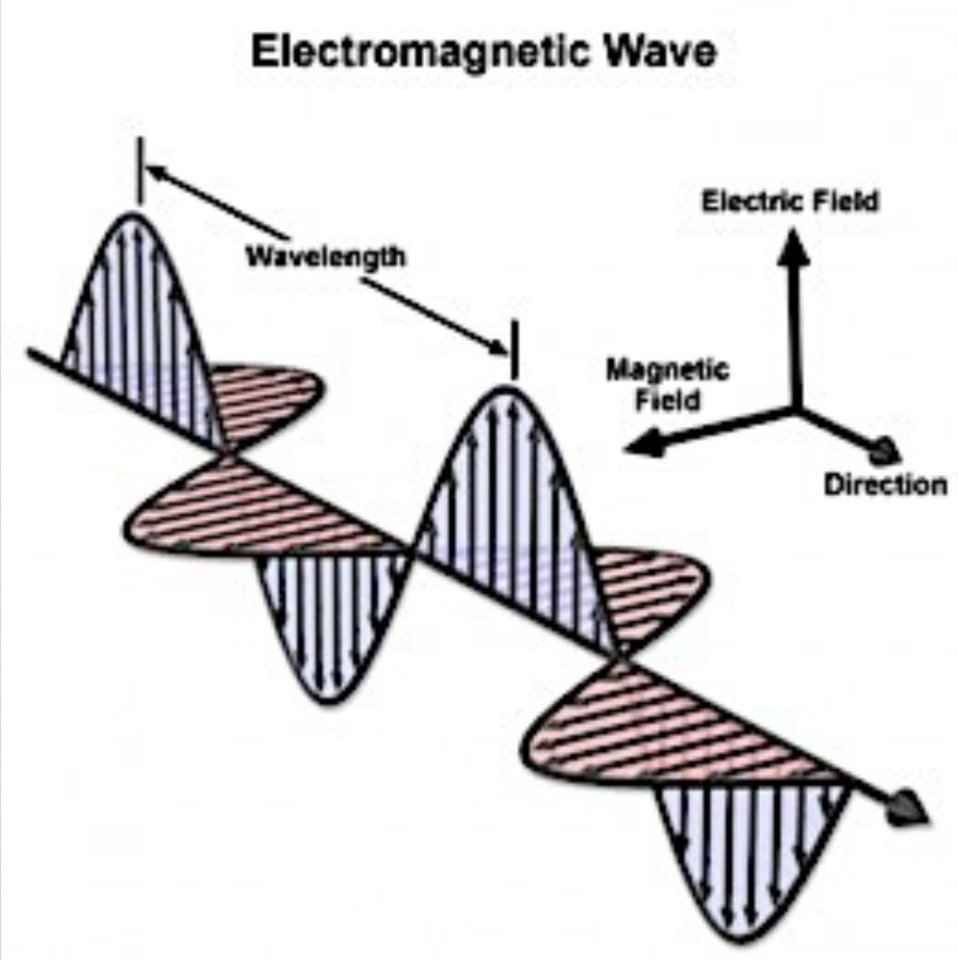
$$\nabla'_x = \cos\theta\nabla_x - \sin\theta\nabla_y \qquad \nabla'_y = \sin\theta\nabla_x + \cos\theta\nabla_y$$

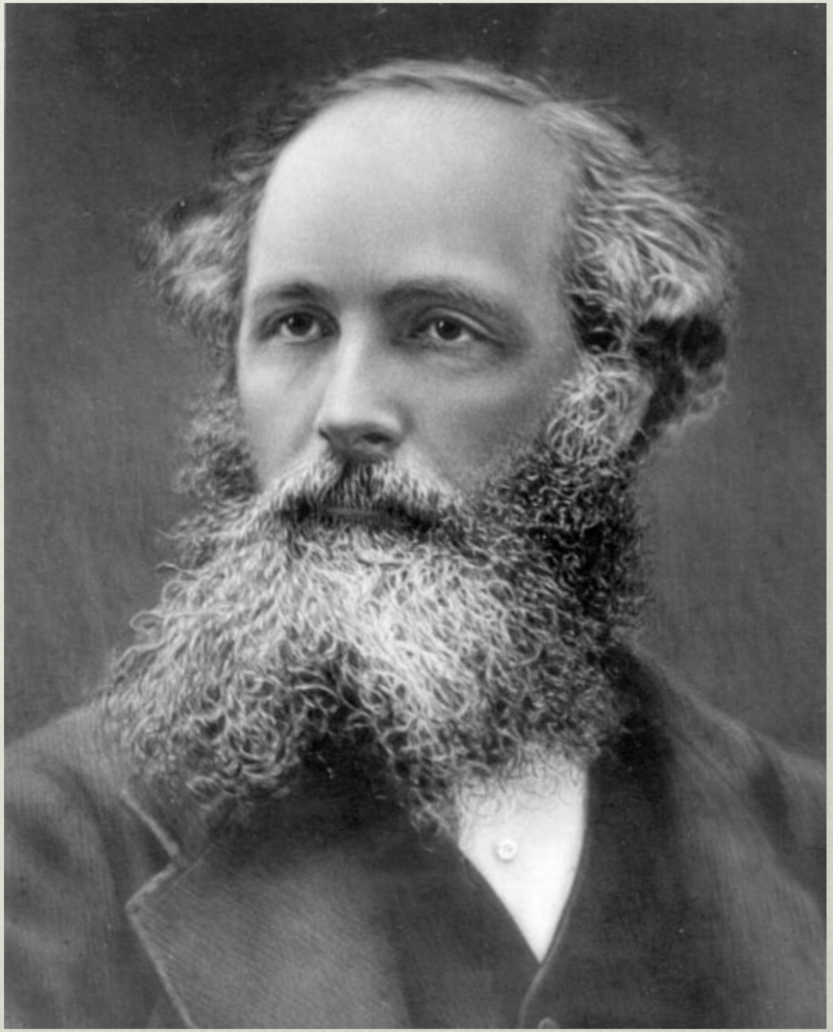
$$\boxed{{\nabla'_x}^2 + {\nabla'_y}^2 + {\nabla'_z}^2 = \nabla_x^2 + \nabla_y^2 + \nabla_z^2}$$

$$\nabla^2\Phi = 4\pi G\rho$$

Preserves its form in Galilean relativity

Electromagnetism

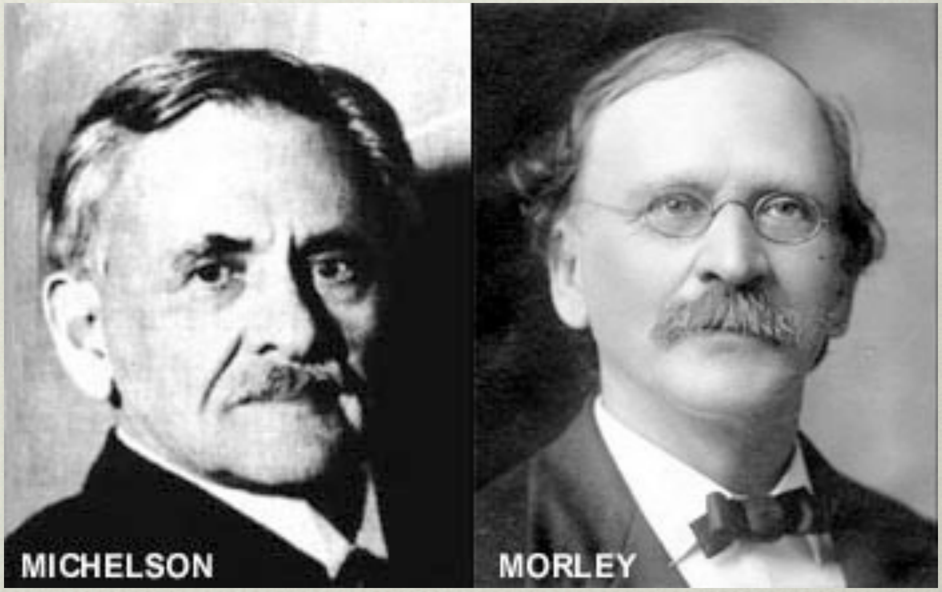




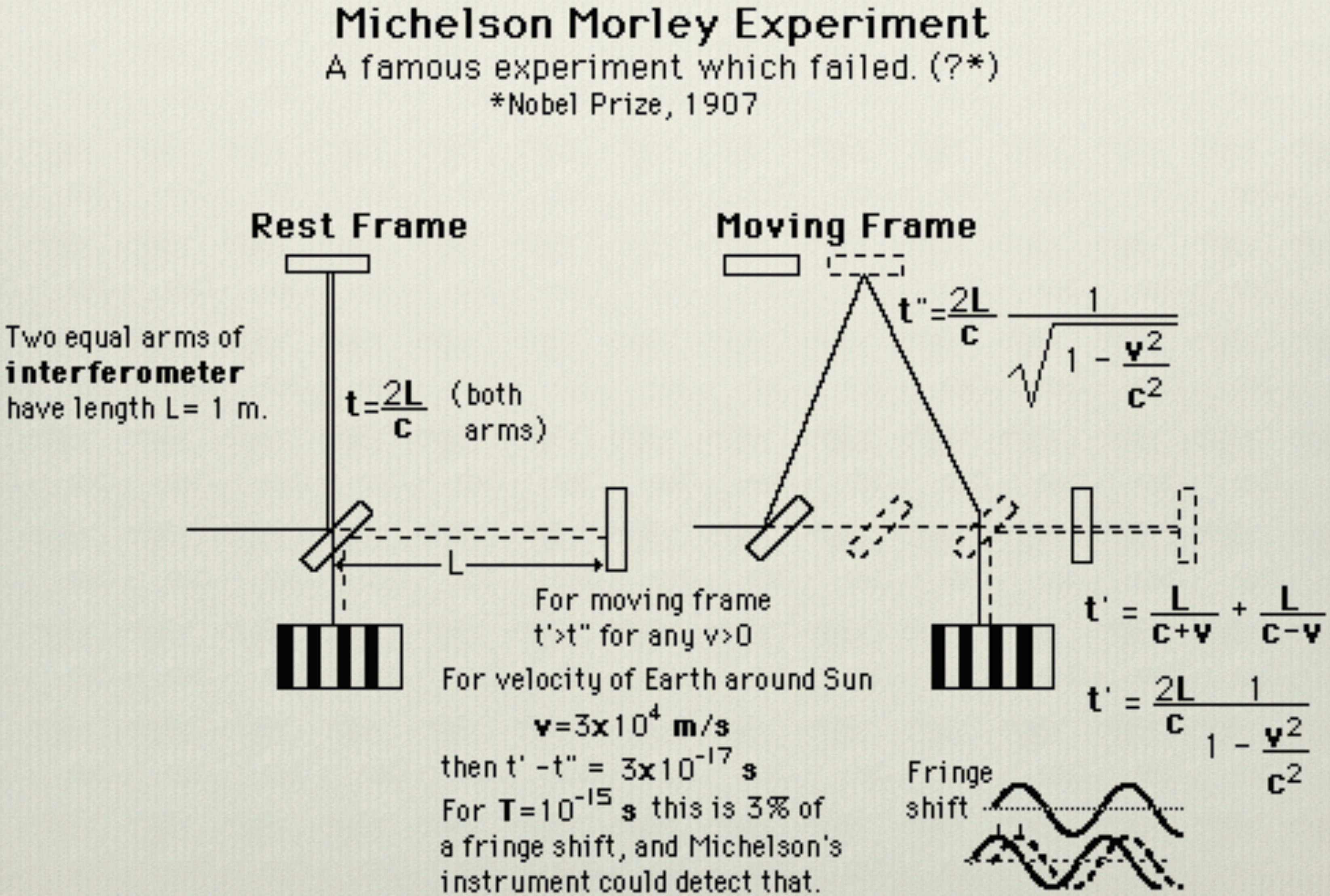
Electromagnetism

- Maxwell showed that light travels with the speed of light
- Galilean transformations: light should travel with different speeds in different inertial frames moving with respect to each other

Albert Michelson and Edward Morley tested the Newtonian addition of velocities "law"



Electromagnetism



Electromagnetism

Maxwell's equations are NOT invariant under Galilean transformations

Maxwell's Equations	Maxwell's Equations
Differential form	Integral form
$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$
$\nabla \times \vec{E} = \boxed{-\frac{\partial \vec{B}}{\partial t}}$	$\oint \vec{E} \cdot d\vec{l} = \boxed{-\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}}$
$\nabla \cdot \vec{B} = 0$	$\oint \vec{B} \cdot d\vec{a} = 0$
$\nabla \times \vec{B} = \mu_0 \vec{J} + \boxed{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$	$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \boxed{\mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}}$

Special Relativity

- The principle of relativity: *Identical experiments carried out in different inertial frames give identical results* was not fulfilled for electromagnetic phenomena when considering Galilean transformations

- Einstein abandoned the idea of absolute time, and derived a new mechanics in which the speed of light is the same in all inertial frames

- The new connection between inertial frames had been derived by Poincare in 1905 to enforce invariance of Maxwell’s equations [he shared the derivation by mail with Lorentz]

Poincare's invariance

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\,\beta_x & -\gamma\,\beta_y & -\gamma\,\beta_z \\ -\gamma\,\beta_x & 1+(\gamma-1)\dfrac{\beta_x^2}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} \\ -\gamma\,\beta_y & (\gamma-1)\dfrac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_y^2}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} \\ -\gamma\,\beta_z & (\gamma-1)\dfrac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\dfrac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_z^2}{\beta^2} \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \beta^i = \frac{\mathbf{v}^i}{c}$$

Notice that: $-c\,\mathrm{d}t'^2 + \mathrm{d}\mathbf{r}'^2 = -c\,\mathrm{d}t^2 + \mathrm{d}\mathbf{r}^2$

Implications of Poincare's invariance

- The line element (invariant in all inertial frames) of flat spacetime (special relativity) is:

$$\mathrm{d}S^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{r}^2$$

- Re-cast the line element using the compact form

$$\mathrm{d}S^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$$

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

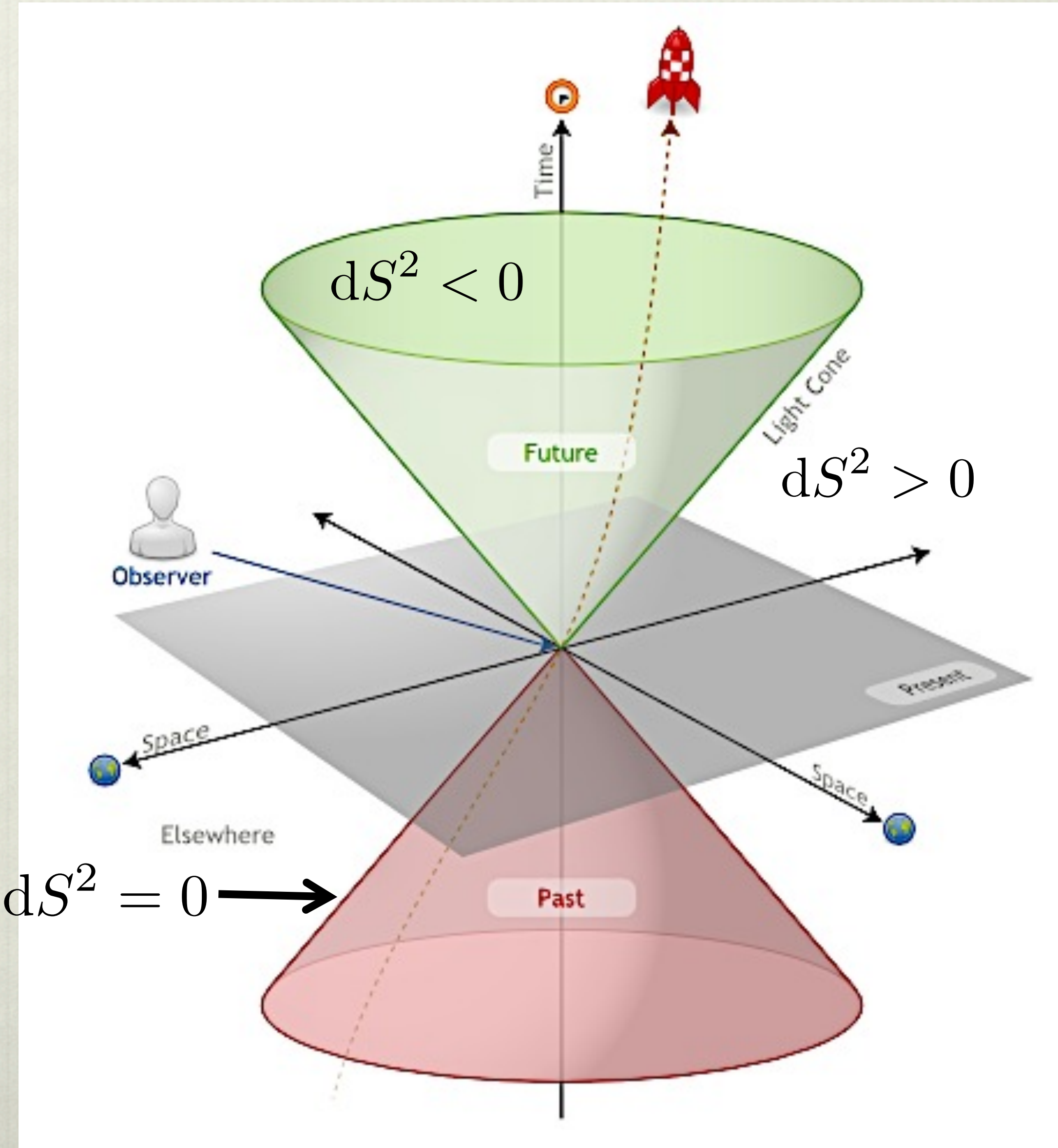
Implications of Poincare's invariance

Geometry of
physical spacetime is
no longer Euclidean!

Flat spacetime is
globally hyperbolic

$$\mathrm{d}S^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{r}^2$$

$\mathrm{d}S^2 > 0$ spacelike events
$\mathrm{d}S^2 = 0$ null events
$\mathrm{d}S^2 < 0$ timelike events



Implications of Poincare's invariance

- Clock: device that measures timelike distances
- Ruler: device that measures spacelike distances
- Proper time: time measured by a clock carried along a particle's worldline

$$c^2\,\mathrm{d}\tau^2 = -\mathrm{d}S^2 \longrightarrow \boxed{\tau_{AB} = \int_{t_A}^{t_B} \mathrm{d}t' \left[1 - \left[\frac{1}{c}\frac{\mathrm{d}\mathbf{r}(\mathbf{t}')}{\mathrm{d}t'}\right]^2\right]^{1/2}}$$

For short Δt and V approximately constant over time:

$$\tau_{AB} = \mathrm{d}t\sqrt{1 - \frac{\mathbf{V}}{c^2}}$$

Valid for accelerating clocks!!

Implications of Poincare's invariance

In rest frame (t, x) the proper length of a rod is

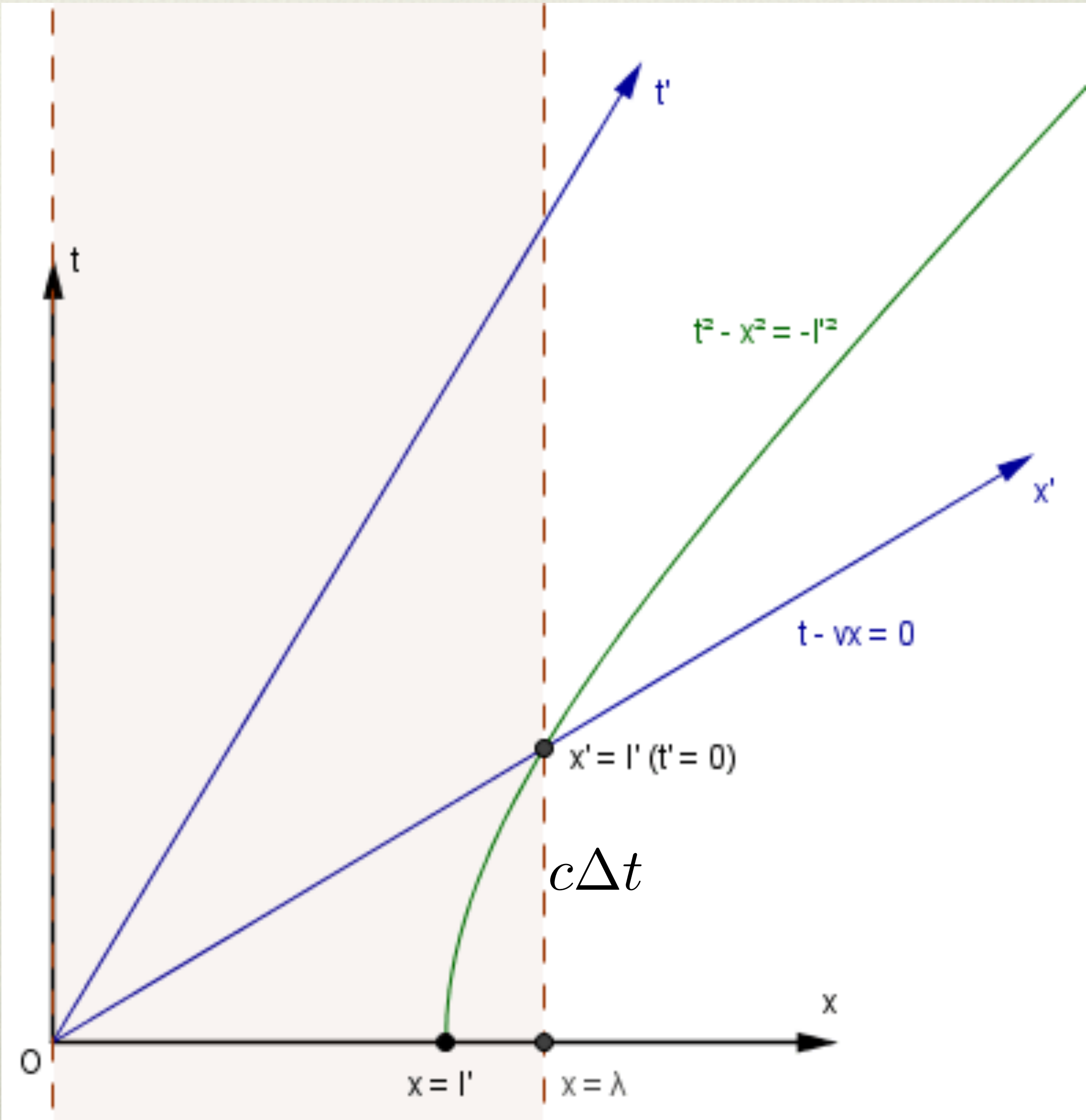
λ

What is its length when measured in an inertial frame which is moving with speed v?

$$-{l'}^2 = t^2 - x^2 \rightarrow (v\lambda)^2 - \lambda^2$$

$$l' = \sqrt{1 - v^2}\lambda$$

Lorentz contraction!



Relativistic Mechanics

Four velocity: four vector whose components are the derivatives of the position along the worldline with respect to the proper time:

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$$

Tangent to the worldline at each point

$$u^t = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-v^2}} \qquad u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{v^i}{\sqrt{1-v^2}}$$

$$u^\alpha = (\gamma, \gamma \mathbf{V}) \longrightarrow u^\alpha u_\alpha = -1$$

Relativistic Mechanics

Energy momentum:

$$p^\alpha = m u^\alpha$$

$$p^t = \frac{m}{\sqrt{1-v^2}} \qquad\qquad p^i = \frac{m v^i}{\sqrt{1-v^2}}$$

$$p^\alpha p_\alpha = -m^2$$

$$p^\alpha = (E, \mathbf{p}) = (m\gamma, m\gamma\mathbf{v}) \longrightarrow E^2 = m^2 + \mathbf{p}^2$$

For a particle at rest:

$$E = mc^2$$

Relativistic Mechanics

Energy momentum for massless particles

In any inertial frame:

$$E = \hbar\omega$$

For the spatial part of the four-momentum:

$$\mathbf{p} = E\mathbf{v} \qquad\qquad |\mathbf{p}| = E$$

Parameterize using wave three vector $\quad \mathbf{p} = \hbar\mathbf{k}$

$$p^\alpha = (E, \mathbf{p}) = (\hbar\omega, \hbar\mathbf{k}) \qquad\qquad p^\alpha p_\alpha = 0$$

Relativistic Mechanics

Assume that a source emits photons of frequency ω in all directions in the source's rest frame. In another inertial frame the source is moving with speed v along the x' axis. What frequency will be observed for a photon that makes an angle α' with the direction of motion?

In S: $k^{\alpha} = (\omega, \mathbf{k})$ In S': $k^{\alpha'} = (\omega', \mathbf{k}')$

Using Lorenz Poincare's transformations $g^t = \gamma\left(g^{t'} - v g^{x'}\right)$

$$\omega = \gamma\left(\omega^{'} - v k^{x'}\right) \text{ with } k^{x'} = \omega' \cos\alpha'$$

Relativistic Doppler shift

$$\boxed{\omega' = \omega \frac{\sqrt{1 - v^2}}{1 - v\cos\alpha'}}$$

General Relativity

The gravitational field has only a relative existence. Because for an observer falling freely from the roof of a house there exists – at least in his immediate surroundings – no gravitational field. Indeed, if the observer drops some bodies then these remain relative to him in a state of rest or uniform motion, independent of their particular chemical or physical nature (in this consideration air resistance is, of course, ignored). The observer has the right to interpret his state as 'at rest'.

General Relativity

- The equivalence principle states that if two bodies initially have the same position and velocity, then they will follow exactly the same trajectory in a gravitational field, even if the have very different composition

- In an electromagnetic field bodies with different charge to mass ratio follow different trajectories

- Trajectories of test bodies in a gravitational field are determined by the structure of spacetime alone. Hence, gravity should be described geometrically

Geodesic Motion

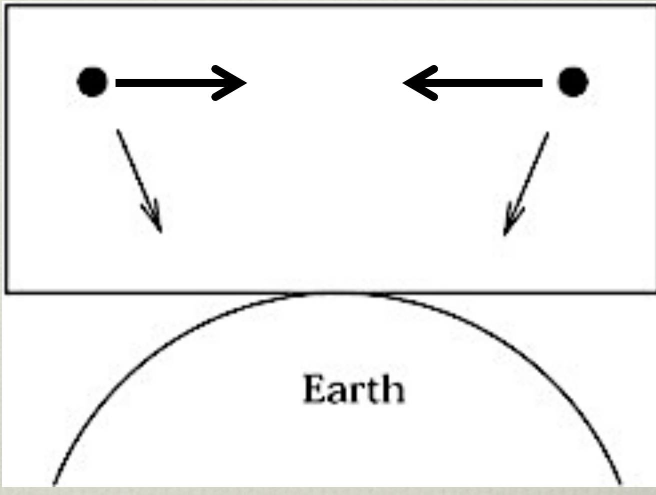
- Why do we need curved spacetime?
- Everywhere the geometry of spacetime is locally Lorentzian: particles move in a straight line with constant velocity… or do they?

Consider two neighboring test particles freely falling radially towards the Earth

The gravitational attraction of the particles is tiny and can be *neglected*

As the particles fall towards the Earth the particles accelerate towards each other as a result of the non-uniformity of the gravitational field

Tidal field *cannot be eliminated by free fall!*



Geodesic Motion

Geodesic: the worldline of a free test particle between two timelike separated points extremizes the proper time between them

$$\tau_{AB} = \int_A^B \mathrm{d}\tau = \int_A^B \left[-\mathrm{d}S^2\right]^{1/2} \xrightarrow[x^\alpha = x^\alpha(\sigma)]{} \tau_{AB} = \int_0^1 \mathrm{d}\sigma \left[-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}\right]^{1/2}$$

The worldliness that extremize proper time between A and B are those that satisfy Lagrange's equations:

$$-\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial(\mathrm{d}x^\alpha/\mathrm{d}\sigma)}\right) + \frac{\partial L}{\partial x^\alpha} = 0$$

$$L\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}, x^\alpha\right) = \left[-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}\right]^{1/2}$$

Geodesic Motion

In an arbitrary spacetime, the equations of geodesic motion are given by:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0 \qquad g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$$

Physical meaning of Christoffel symbols?

Consider the Newtonian limit: the gravitational field is static and velocities are small:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}t}{\mathrm{d}\tau}$$

The geodesic equation becomes:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} - \frac{1}{2}g^{\alpha\delta}\partial_\delta g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0$$

Geodesic Motion

Write the metric as a small deviation from flat spacetime:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

Geodesic equation takes the form:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = \frac{1}{2}\eta^{\alpha\delta}\partial_\delta h_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

Time component:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0$$

Spatial component:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \frac{1}{2}\partial_i h_{tt}$$

Geodesic Motion

Compare these expressions to Newton's equations of motion

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\nabla\Phi$$

Hence, in the weak-field limit:

$$g_{tt} = -\left(1 + 2\Phi\right), \quad g_{ti} = g_{it} = 0, \quad g_{ij} = \delta_{ij}$$

Hence, Newtonian equations of motion can be re-cast as:

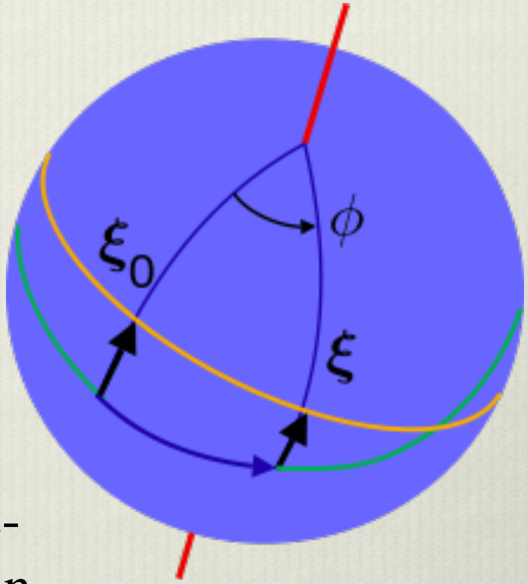
$$\nabla^2\Phi = 4\pi G\rho \rightarrow \nabla^2 h_{tt} = -8\pi G T_{tt}$$

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + \eta^{\alpha\beta} p$$

Geodesic Motion

The separation of two nearby geodesics satisfies the equation

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + R\xi = 0$$



R is the Gaussian curvature of the surface: it changes from place to place

This relation is valid in 2-D as long as the separation vector ξ remains perpendicular to the blue-green geodesic

In higher dimensions the separation vector can remain perpendicular to the blue-green geodesic but rotate about it: we need to specify not only its magnitude but also its direction

Geodesic Motion

In arbitrary space-time the deviation of two nearby geodesics with tangent vector vectors $\mathbf{u} = u^\alpha \mathbf{e}_\alpha$ and separation vector $\mathbf{n} = n^\alpha \mathbf{e}_\alpha$ is determined by the Riemann tensor:

$$\frac{D^2 n^\alpha}{\mathrm{d}u^2} + R^\alpha_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0$$

The Riemann tensor is *the embodiment of the bends and warps of space-time*:

$$R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\sigma_{\beta\delta} - \Gamma^\alpha_{\sigma\delta}\Gamma^\sigma_{\beta\gamma}$$

The intrinsic derivative of a vector along a curve x^γ is defined as:

$$\frac{Dv^\alpha}{\mathrm{d}u} = \frac{\mathrm{d}v^\alpha}{\mathrm{d}u} + \Gamma^\alpha_{\beta\gamma} v^\beta \frac{\mathrm{d}x^\gamma}{\mathrm{d}u}$$

Riemann Tensor

The additional objects can be derived from the Riemann tensor:

Ricci tensor : $R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}$ Ricci scalar : $R = g^{\alpha\beta} R_{\alpha\beta}$

The Ricci tensor and Ricci scalar are also related using the covariant derivative, which is defined as:

$$\nabla_{\alpha} v^{\beta} = \partial_{\beta} v^{\alpha} + \Gamma^{\alpha}_{\sigma\beta} v^{\beta}$$

The covariant derivative of the Ricci tensor is given by

$$\nabla^{\alpha} R_{\alpha\beta} = \frac{1}{2} \nabla_{\beta} R$$

Einstein Tensor

Gathering the above results, define the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

Note that since $\nabla \mathbf{g} = \mathbf{0}$, then $\nabla \mathbf{G} = 0$

Field Equations

Consider again Newtonian field equations:

$$\nabla^2\Phi = 4\pi G\rho \rightarrow \nabla^2 h_{tt} = -8\pi G T_{tt}$$

The geometric object that generalizes these equations in curved spacetime should contain second order derivatives of the metric tensor. Note that since the right hand side of Newton's field equations involves the energy momentum tensor, and matter fields satisfy energy conservation then

$$\nabla^\alpha T_{\alpha\beta} = 0$$

The geometric object that describes gravity should also have vanishing covariant derivative.

Field Equations

The above constrains are satisfied by the Einstein tensor $G_{\alpha\beta}$. Hence consider $G_{\alpha\beta} = \kappa T_{\alpha\beta}$ and explore the weak-field limit of these equations:

$$R_{tt} = -\frac{1}{2}\nabla^2 h_{tt}$$

Relativistic and Newtonian field equations take the form:

$$R_{tt} - \frac{1}{2}R = \kappa T_{tt} \to R_{tt} = \frac{1}{2}\kappa T_{tt}$$

Finally, notice that:

$$\nabla^2 h_{tt} = -\kappa T_{tt} \to \kappa = 8\pi G$$

The equations describing gravity in general relativity are:

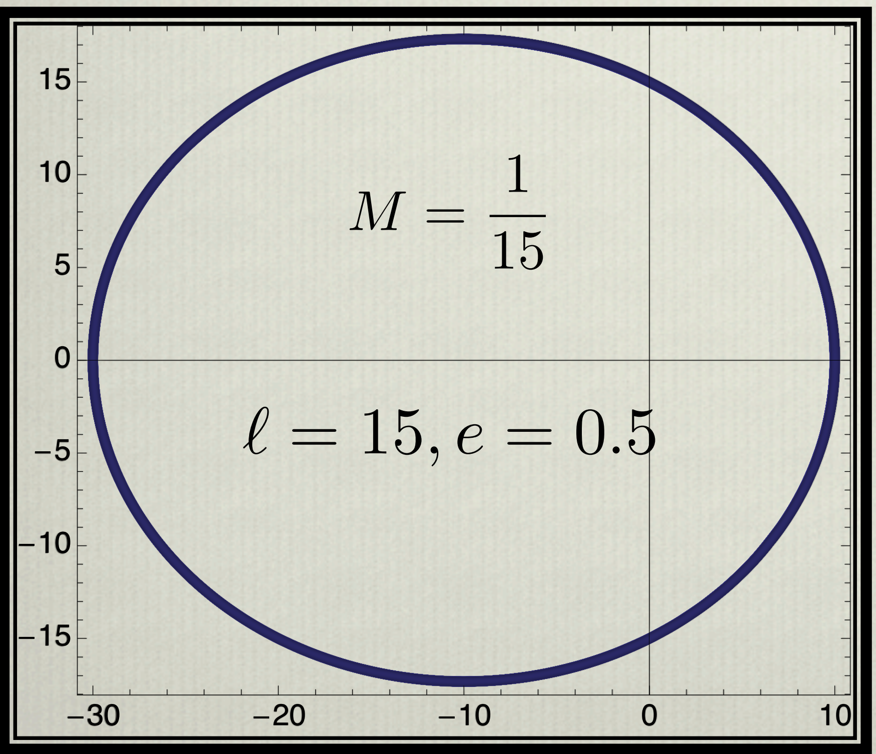
$$G_{\alpha\beta} = 8\pi G\, T_{\alpha\beta}$$

Schwarzschild geometry

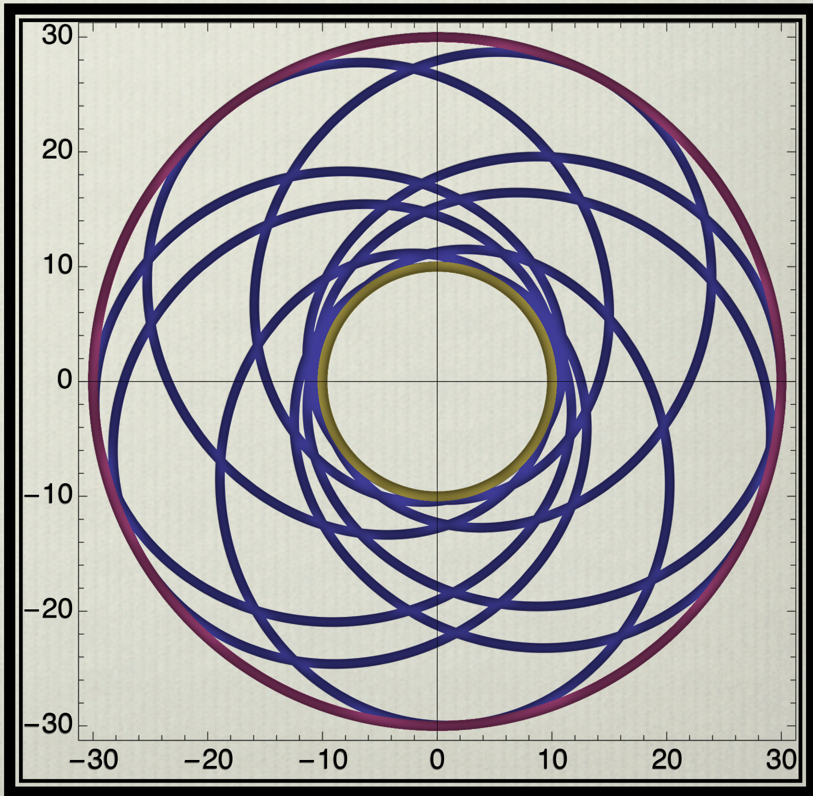
Describes the empty spacetime outside a spherically symmetric source of curvature. Line element is:

$$\mathrm{d}S^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

Newtonian Ellipse



Relativistic orbit



Predictions

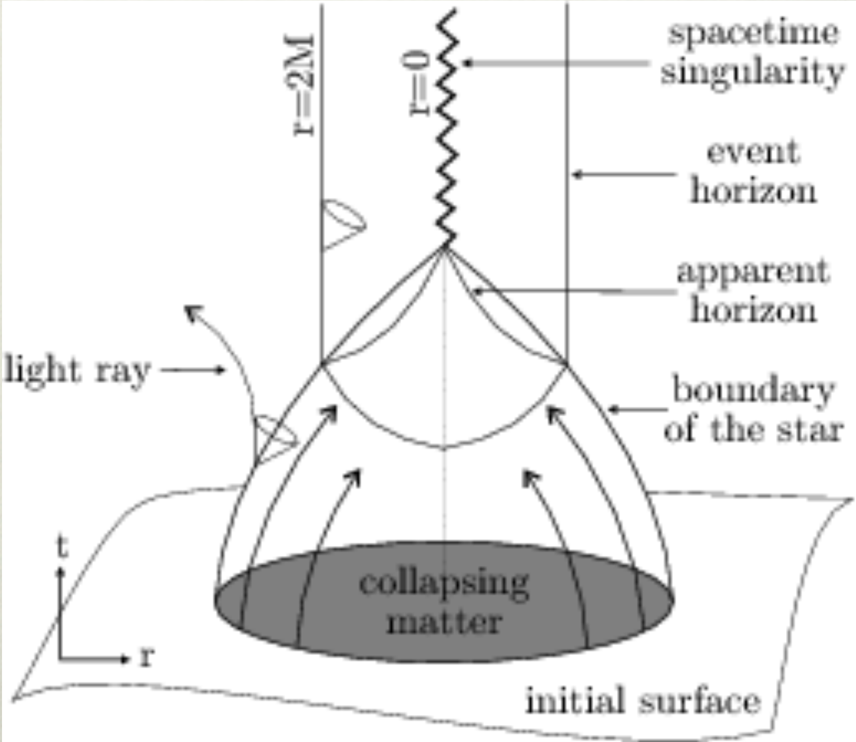
- Perihelion precession: $$\delta\phi = \frac{6\pi GM}{ac^2\left(1-e^2\right)}$$

- Formation of compact objects

Collapsing sphere of dust whose forming particles are freely falling following radial geodesics in the Schwarzschild geometry

The sphere reaches the horizon in a finite amount of proper time

Once inside the horizon, it hits the singularity in a proper time 4M/3



Astrophysical Black Holes

The most general stationary black hole solutions of the vacuum Einstein equations are described by a family of geometries, discovered by Roy Kerr in 1963, that depend on two parameters: the total mass [M] and total angular momentum [J] of the black hole

$$\boxed{\begin{aligned} \mathrm{d}S^2 &= -\left(1-\frac{2Mr}{\Sigma^2}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma^2}\mathrm{d}\phi\mathrm{d}t + \frac{\Sigma^2}{\Delta}\mathrm{d}r^2 \\ &+ \Sigma^2\mathrm{d}\theta^2 + \left(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\Sigma^2}\right)\sin^2\theta\mathrm{d}\phi^2 \end{aligned}}$$

$$a = J/M,$$
$$\Sigma^2 = r^2 + a^2\cos^2\theta,$$
$$\Delta = r^2 - 2Mr + a^2$$

Hot iron atoms produce a strong signature of X-rays at a specific energy, which is smeared out by the rotation of the black hole. The nature of this smearing can then be used to infer the spin rate.

Gravitational Waves

In a region of space that is nearly flat:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll 1$$

Use normal coordinates where $\Gamma^{\alpha}_{\beta\gamma} = 0$, and expand Einstein equations to obtain:

$$R_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta,\gamma} - \Gamma^{\gamma}_{\alpha\gamma,\beta} = \frac{1}{2}\left(h_{\alpha}{}^{\gamma}{}_{,\beta\gamma} + h_{\beta}{}^{\gamma}{}_{,\alpha\gamma} - h_{\alpha\beta,\gamma}{}^{\gamma} - h_{,\alpha\beta}\right), \quad h = \eta^{\alpha\beta}h_{\alpha\beta}$$

Using $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$, the linearized equations take the form

$$-\bar{h}_{\alpha\beta,\gamma}{}^{\gamma} - \eta_{\alpha\beta}\bar{h}_{\gamma\delta,}{}^{\gamma\delta} + \bar{h}_{\alpha\gamma,}{}^{\gamma}{}_{\beta} + \bar{h}_{\beta\gamma,}{}^{\gamma}{}_{\alpha} = 16\pi T_{\alpha\beta}$$

Gravitational Waves

Consider an infinitesimal coordinate transformation (also called gauge transformation)

$$x^{'\alpha} = x^{\alpha} + \xi^{\alpha}$$

Then the linearized metric takes the form

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

Choose the Lorenz gauge $\Box\xi^{\alpha} = \bar{h}^{\alpha\beta}{}_{,\beta}$ so that $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$. In this gauge, the linearized equations reduce to

$$\Box\bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}$$

Gravitational Waves

The solution to this wave equation is

$$\bar{h}_{\alpha\beta} = 4\int_V \frac{T_{\alpha\beta}\left(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}V' \to \bar{h}_{\alpha\beta} = \mathcal{R}\left[W_{\alpha\beta}e^{i\kappa_\sigma x^\sigma}\right]$$

Use residual gauge freedom to impose two additional constraints:

1. For a uniform velocity field u^α impose $W_{\alpha\beta}u^\alpha = 0$,
2. $\eta^{\alpha\beta}W_{\alpha\beta} = 0$

These two conditions are known as the transverse traceless gauge. Since any gravitational wave can be described as a superposition of plane waves, and it is always possible to find a gauge to re-cast every plane wave in this gauge (linearity), then all gravitational waves are transverse and traceless!

Beyond the linear regime

Using the Lorenz gauge and incorporating perturbations at order $\mathcal{O}(h^2)$ and beyond, gravitational waves can be described in an arbitrary space-time as follows

$$\bar{h}_{\alpha\beta} = 4\int_V \frac{\left(T_{\alpha\beta} + t_{\alpha\beta}\right)\left(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|\right)}{|\mathbf{x} - \mathbf{x}'|}\mathrm{d}V'$$

where $t_{\alpha\beta}$ in the gravitational stress energy. Note that:

$$\int_V \left(T_{ij} + t_{ij}\right)\mathrm{d}V' = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_{ij}(t) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int \left(T_{tt} + t_{tt}\right) x'^i x'^j \mathrm{d}V'\right]$$

Hence, an exact solution to the vacuum field equations is

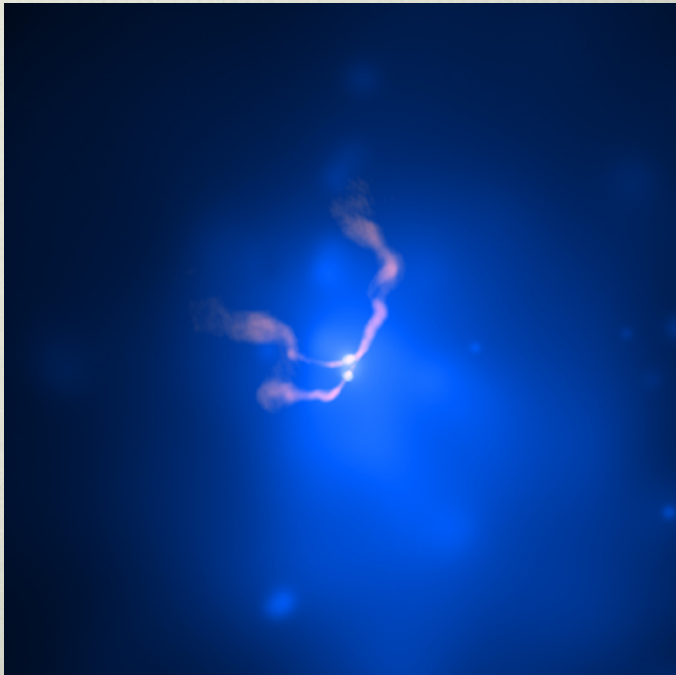
$$h_{ij}\left(\mathbf{x}, t\right) = \frac{2}{|\mathbf{x}|}\ddot{I}_{ij}\left(t - \mathbf{x}\right)$$

Gravitational Waves

Gravitational waves carry energy and angular momentum from a radiating system:

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \int T^{tr} r^2 \mathrm{d}\Omega = \frac{1}{5} < \dddot{I}_{ij}(t-r)\,\dddot{I}^{ij}(t-r) >, \\
\frac{\mathrm{d}L_i}{\mathrm{d}t} &= \int \epsilon_{ijk} x_j T_{kr} r^2 \mathrm{d}\Omega = \frac{2}{5} < \epsilon_{ijk} \ddot{I}^{jl}(t-r)\,\dddot{I}^{k}{}_{l}(t-r) >
\end{aligned}
$$

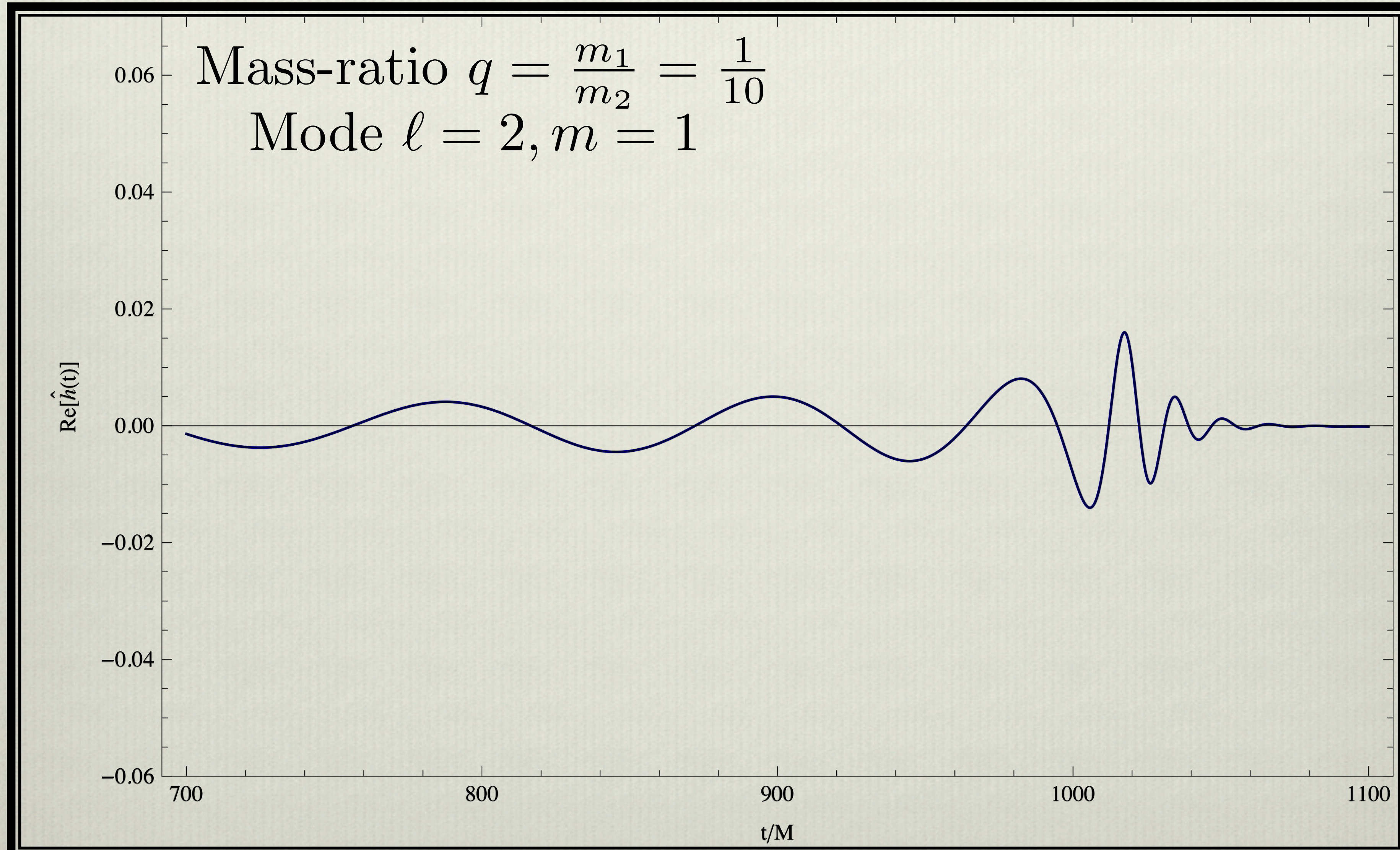
Two supermassive black holes spiral towards each other near the center of a galaxy cluster named Abell 400. Shown in this X-ray/radio composite image are the multi-million degree radio jets emanating from the black holes. Credit: X-ray: NASA/CXC/AIfA/D.Hudson



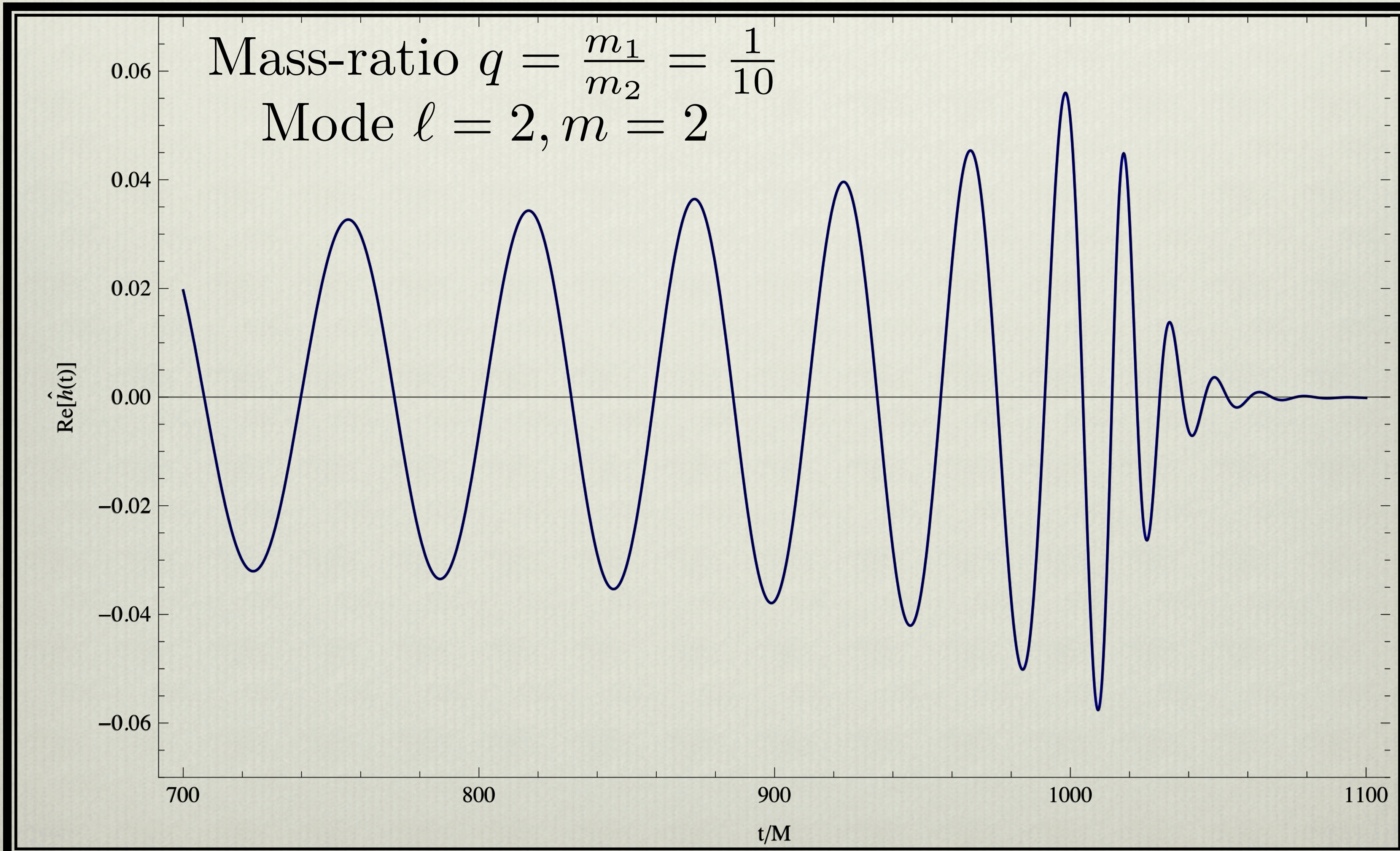
Gravitational Waves

- Fluctuations in curvature that propagate at the speed of light
- Produced by changes in the quadrupole moment of a mass distribution: no monopole or dipole gravitational waves
- Are transverse and traceless
- Carry energy and angular momentum lost by radiation reaction in a radiating source
- Can be lensed and red-shifted

Rogues Gallery



Rogues Gallery



Rogues Gallery